On the Constructively Determination the Spectral Invariants of the Periodic Multidimensional Schrödinger Operator

O. A. Veliev
Depart. of Math., Faculty of Arts and Sci., Dogus University,
Acibadem, Kadikoy, Istanbul, Turkey,
e-mail: oveliev@dogus.edu.tr

Abstract

In this paper we constructively determine a family of the spectral invariants of the multidimensional Schrödinger operator with a periodic potential by the given band functions.

1 Introduction

We investigate the Schrödinger operator

$$L(q) = -\Delta + q(x),\ x \in \mathbb{R}^d,\ d \geq 2 \tag{1}$$

with a real periodic (relative to the lattice Ω) potential $q(x) \in W_2^s(F)$, where $s \geq 6(3^d(d+1)^2)+d$ and F is the fundamental domain $\mathbb{R}^d/\Omega$ of Ω. The spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$ for $t \in F^* \equiv \mathbb{R}^d/\Gamma$ generated by (1) and the conditions

$$u(x+\omega) = e^{i(t,\omega)}u(x),\ \forall \omega \in \Omega,$$

where $\Gamma \equiv \{\delta \in \mathbb{R}^d : (\delta,\omega) \in 2\pi\mathbb{Z}, \forall \omega \in \Omega\}$ is the lattice dual to Ω (see [1]). The eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq$ ...of $L_t(q)$ define functions $\Lambda_1(t)$, $\Lambda_2(t), ...,$ of t that are called the band functions of $L(q)$. In this paper using the asymptotic formulas for the band functions and the Bloch functions obtained in [4], we obtain more detailed asymptotic formulas and then constructively determine a family of the spectral invariants by the given band functions. In introduction we list the main results. In section 2 we prove the main results without giving some estimations which are given in section 3 and in appendices.

Let δ be a maximal element of Γ, that is, δ is the nonzero element of Γ of minimal norm belonging to the line $\delta\mathbb{R}$ and

$$q^\delta(x) = \sum_{n \in Z} q_{n\delta} e^{in(\delta,x)} = Q(\zeta) \tag{2}$$

be the directional (one dimensional) potential, where $\zeta = (\delta, x)$ and

$$q_\gamma = (q(x), e^{i(\gamma,x)}) = \int_F q(x) e^{-i(\gamma,x)} dx$$

is the Fourier coefficient of $q(x)$. Without loss of generality we assume that the measure $\mu(F)$ of F is 1 and $q_0 = 0$. Let $\lambda_0 \leq \lambda_2^- \leq \lambda_2^+ ...$ and

$\lambda_1^- \leq \lambda_1^+ \leq \lambda_3^- \leq \lambda_3^+ ...$be the eigenvalues of the boundary value problem

$$- \mid \delta \mid^2 y''(\zeta) + Q(\zeta) y(\zeta) = \mu y(\zeta), \; y(\zeta + 2\pi) = e^{i2\pi v} y(\zeta) \tag{3}$$

for $v = 0$ and $v = \frac{1}{2}$ respectively, where $\mid \delta \mid$ is the norm of δ. The corresponding eigenfunctions are denoted by $\varphi_0(s)$ and $\varphi_n^{\pm}(s)$ respectively.

In the pioneering paper [2] about isospectral potentials it was proved that if $q(x) \in C^6(F)$, $\omega \in \Omega \backslash 0$, and δ is the maximal element of Γ satisfying $(\delta, \omega) = 0$ then given band functions one may recover $\lambda_0, \lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+, ...$ and

$$\int_F \mid Q_\omega(x) \varphi_n^{\pm}(s) \mid^2 dx \;\; \text{if} \;\; \lambda_n^- < \lambda_n^+,$$

or $\int_F \mid Q_\omega(x) \mid^2 ((\varphi_n^+(s))^2 + (\varphi_n^-(s))^2) dx$ if $\lambda_n^- = \lambda_n^+$, where

$$Q_\omega(x) = \sum_{\gamma : \gamma \in \Gamma, (\gamma,\omega) \neq 0} \frac{\gamma}{(\omega, \gamma)} q_\gamma e^{i(\gamma,x)}.$$

The proofs given there were nonconstructive. In paper [3] it was given a constructive way of determining the spectrum of $L_t(q^\delta)$ from the spectrum of $L_t(q)$ for the two dimensional ($d = 2$) case (see remark (1) of [3]).

In this paper, for arbitrary dimension d, by the given band functions we constructively determine the all eigenvalues of the boundary value problem (3) for all values of v and a family of new spectral invariants

$$J(\delta, b, n, v), \; J_0(\delta, b), \; J_1(\delta, b), \; J_2(\delta, b) \quad \text{(see (12), (15))}$$

for $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $n \in \mathbb{Z}$, $\delta \in M(\Gamma)$, $b \in M(\Gamma_\delta)$, where $M(\Gamma)$ and $M(\Gamma_\delta)$ are the set of all maximal elements of the lattices Γ and Γ_δ respectively, Γ_δ is the dual lattice of Ω_δ and $\Omega_\delta = \{h \in \Omega : (h, \delta) = 0\}$ is the sublattice of Ω in the hyperplane $H_\delta = \{x \in \mathbb{R}^d : (x, \delta) = 0\}$. Note that $J_k(\delta, b)$ is explicitly expressed by Fourier coefficient of $q(x)$. Moreover, if $d > 2$ and $q(x)$ is a trigonometric polynomial then, in general, the number of nonzero spectral invariants $J_k(\delta, b)$ is greater than the number of nonzero Fourier coefficient of $q(x)$.This situation allows us to give (it will be given in next papers) an algorithm for finding the potential $q(x)$ from these spectral invariants.

Let us describe the brief scheme of this paper. First using the asymptotic formulas for the band functions and the Bloch functions obtained in [4], we obtain more detailed asymptotic formulas and then using these formulas we

constructively determine the family of the spectral invariants. The eigenvalues of the operator $L_t(0)$ with zero potential are $\mid \gamma + t \mid^2$ for $\gamma \in \Gamma$. If the quasimomentum $\gamma + t$ lies near the diffraction plane

$$D_\delta = \{x \in \mathbb{R}^d : \mid x \mid^2 - \mid x + \delta \mid^2 = 0\}, \tag{4}$$

then the corresponding eigenvalue of $L_t(q)$ is close to the eigenvalue of the operator $L_t(q^\delta)$ with directional potential (2). To describe the eigenvalue of $L_t(q^\delta)$ we consider the lattice Γ_δ. Let $F_\delta \equiv H_\delta / \Gamma_\delta$ be the fundamental domain of Γ_δ. In this notation the quasimomentum $\gamma + t$ has the orthogonal decompositions

$$\gamma + t = \beta + \tau + (j + v)\delta, \tag{5}$$

where $\beta \in \Gamma_\delta \subset H_\delta$, $\tau \in F_\delta \subset H_\delta$, $j \in \mathbb{Z}$, $v \in [0, 1)$ and v depends on β and t. The eigenvalues and eigenfunctions of the operator $L_t(q^\delta)$ are

$$\lambda_{j,\beta}(v, \tau) = \mid \beta + \tau \mid^2 + \mu_j(v), \ \Phi_{j,\beta}(x) = e^{i(\beta+\tau,x)} \varphi_{j,v}(\zeta) \tag{6}$$

for $j \in \mathbb{Z}$, $\beta \in \Gamma_\delta$, where $\mu_j(v)$ and $\varphi_{j,v}(\zeta)$ are eigenvalues and eigenfunctions of the operator $T_v(Q)$ generated by the boundary value problem (3). We say that the large quasimomentum (5) lies near the diffraction plane (4) if

$$\frac{1}{2}\rho < \mid \beta \mid < \frac{3}{2}\rho, \ j = O(\rho^{\alpha_1}), \tag{7}$$

where ρ is large parameter, $\alpha = \frac{1}{4(3^d(d+1))}$, and $\alpha_k = 3^k \alpha$ for $k = 1, 2, ..., d$. In this paper we construct a set of quasimumentum near the diffraction plane D_δ such that if $\beta + \tau + (j + v)\delta$ (see (5)) belongs to this set, then there exists a simple eigenvalue, denoted by $\Lambda_{j,\beta}(v, \tau)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v, \tau) = \lambda_{j,\beta}(v, \tau) + O(\rho^{-a}), \tag{8}$$

$$\Lambda_{j,\beta}(v, \tau) = \lambda_{j,\beta}(v, \tau) + \frac{1}{4}\int_F \mid f^2_{\delta,\beta+\tau} \mid |\varphi_{j,v}|^2 \, dx + O(\rho^{-3a+2\alpha_1} \ln \rho), \tag{9}$$

where $a = 1 - \alpha_d + \alpha$ and

$$f_{\delta,\beta+\tau}(x) = \sum_{\gamma:\gamma\in\Gamma\backslash\delta\mathbb{R}, |\gamma|<\rho^\alpha} \frac{\gamma}{(\beta + \tau, \gamma)} q_\gamma e^{i(\gamma,x)}.$$

The eigenfunction $\Psi_{j,\beta}(x)$ corresponding to $\Lambda_{j,\beta}(v, \tau)$ satisfies

$$\Psi_{j,\beta}(x) = \Phi_{j,\beta}(x) + O(\rho^{-a}). \tag{10}$$

Besides we prove that derivative of $\Lambda_{j,\beta}(v, \tau)$ in direction $h = \frac{\beta+\tau}{|\beta+\tau|}$ satisfies

$$\mid \beta + \tau \mid \frac{\partial \Lambda_{j,\beta}(v, \tau)}{\partial h} = \mid \beta + \tau \mid^2 + O(\rho^{2-2a}) \tag{11}$$

and the derivative of other simple eigenvalues, neighboring with $\Lambda_{j,\beta}(v,\tau)$, does not satisfy (11). Using this formulas we constructively determine the eigenvalues $\mu_n(v)$ for $n \in \mathbb{Z}$, $v \in [0,1)$ and the spectral invariants

$$J(\delta, b, n, v) = \int_F \mid q_{\delta,b}(x)\varphi_{n,v}(\delta, x) \mid^2 dx \tag{12}$$

for $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $n \in \mathbb{Z}$, and for all maximal elements b of Γ_δ, where δ is any maximal element of Γ,

$$q_{\delta,b}(x) = \sum_{\gamma \in S(\delta,b) \backslash \delta\mathbb{R}} \frac{\gamma}{(b,\gamma)} q_\gamma e^{i(\gamma,x)}, \tag{13}$$

$S(\delta, b) = P(\delta, b) \cap \Gamma$, and $P(\delta, b)$ is the plane containing δ, b and 0. Then substituting the asymptotic decomposition

$$|\varphi_{n,v}(\zeta)|^2 = A_0 + \frac{A_1(\zeta)}{n} + \frac{A_2(\zeta)}{n^2} + ..., \tag{14}$$

where $A_k(\zeta)$ is expressed via $Q(\zeta)$ (see (2)), into (12) we find the invariants

$$J_k(\delta, b) = \int_F |q_{\delta,b}(x)|^2 A_k(\zeta) dx \tag{15}$$

for $k = 0, 1, 2, ...$Using the well-known asymptotic formulas for eigenvalues and eigenfunctions of the Sturm-Liouville operator $T_v(Q)$ by direct calculations we find $A_0(\zeta)$, $A_1(\zeta)$, $A_2(\zeta)$ and the invariants

$$\int_F |q_{\delta,b}(x)|^2 q^\delta(x) dx, \tag{16}$$

$$\int_F \left|q^\delta(x)\right|^2 dx \tag{17}$$

(see Appendix D). If the potential $q(x)$ is a trigonometric polynomial then the most of the directional potentials has the form

$$q^\delta(x) = q_\delta e^{i(\delta,x)} + q_{-\delta} e^{-i(\delta,x)}. \tag{18}$$

In this case, by direct calculations, we show that (see Appendix D)

$$\begin{aligned} &A_0 = 1,\ A_1 = 0,\ A_2 = \frac{q^\delta(x)}{2} + a_1 |q_\delta|^2, A_3 = a_2 q^\delta(x) + a_3 |q_\delta|^2, \\ &A_4 = a_4 q^\delta(x) + a_5 (q_\delta^2 e^{i2(\delta,x)} + q_{-\delta}^2 e^{-i2(\delta,x)}) + a_6, \end{aligned} \tag{19}$$

where $a_1, a_2, ..., a_6$ are the known constants. Moreover using (19), (17), and (15) for $k = 2, 4$ we find the invariant

$$\int |q_{\delta,b}(x)|^2 (q_\delta^2 e^{i2(\delta,x)} + q_{-\delta}^2 e^{-i2(\delta,x)}) dx \tag{20}$$

in the case (18). In next paper we give an algorithm for finding the potential $q(x)$ by the invariants (16), (17), and (20).

2 The Proofs of the Main Results

In this section we give the proofs of the main results without getting the technical details. The technical details, namely the proof of lemmas and some estimations are investigated in Sections 3 and in appendices respectively. First let us prove (8). To obtain the asymptotic formulas for large eigenvalues we introduce a large parameter ρ. If the considered eigenvalue is of order ρ^2 we write the potential $q(x) \in W_2^s(F)$ in the form

$$q(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma,x)} + O(\rho^{-p\alpha}), \tag{21}$$

where $p = s - d$, $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 <\mid \gamma \mid< \rho^\alpha)\}$ and α is defined in (7). Note that the relation $q(x) \in W_2^s(F)$ means that

$$\sum_{\gamma \in \Gamma} \mid q_\gamma \mid^2 (1+ \mid \gamma \mid^{2s}) < \infty.$$

This implies that if $s \geq d$, then

$$\sum_{\gamma \in \Gamma} \mid q_\gamma \mid< c_1, \ \sup \mid \sum_{\gamma \notin \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma,x)} \mid \leq \sum_{\mid\gamma\mid \geq \rho^\alpha} \mid q_\gamma \mid = O(\rho^{-p\alpha}), \tag{22}$$

i.e., (21) holds. Here and in subsequent estimations we denote by c_i $(i = 1, 2, ...)$ the positive, independent of ρ, constants.

In [4] (see Theorem 3.1, Theorem 6.1, and (6.45) of [4]) we proved that if

$$j \in S_1(\rho), \ \beta \in S_2(\rho), \ v \in S_3(\beta, \rho), \tau \in S_4(\beta, j, v, \rho), \tag{23}$$

then there exists unique simple eigenvalue $\Lambda_N(t)$, denoted in [4] by $\Lambda(\lambda_{j,\beta})$ and in this paper by $\Lambda_{j,\beta}(v, \tau)$ or by $\Lambda_{N(j,\beta)}(t)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v, \tau) = \lambda_{j,\beta}(v, \tau) + O(\rho^{-\alpha_2}) \tag{24}$$

and the corresponding eigenfunction $\Psi_{N,t}(x)$, denoted here by $\Psi_{j,\beta}(x)$, satisfies

$$\Psi_{j,\beta}(x) = \Phi_{j,\beta}(x) + O(\rho^{-\alpha_2} \ln \rho), \tag{25}$$

where α_2 is defined in (7) and the sets S_k for $k = 1, 2, 3, 4$ are defined as follows:

$$S_1(\rho) = \{j \in \mathbb{Z} :\mid j \mid< \frac{\rho^{\alpha_1}}{2 \mid \delta \mid^2} - \frac{3}{2}\}, \tag{26}$$

$$S_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in (R_\delta(\frac{3}{2}\rho - d_\delta - 1)\backslash R_\delta(\frac{1}{2}\rho + d_\delta + 1))\backslash(\bigcup_{b \in \Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(\rho^{\frac{1}{2}}))\},$$

where $d_\delta = \sup_{x,y \in F_\delta} \mid x - y \mid$ is the diameter of F_δ,

$$R_\delta(c) = \{x \in H_\delta :\mid x \mid< c\}, \ \Gamma_\delta(c) = \{b \in \Gamma_\delta : 0 <\mid b \mid< c\},$$

$$V_b^{\delta}(c) = \{x \in H_\delta :\mid\mid x+b \mid^2 - \mid x \mid^2 \mid < c\},$$

$$S_3(\beta,\rho) = W(\rho)\backslash A(\beta,\rho), \tag{27}$$

where $W(\rho) \equiv \{v \in (0,1) :\mid \mu_j(v) - \mu_{j'}(v) \mid > \frac{2}{\ln \rho}, \forall j^{'}, j \in \mathbb{Z}, j^{'} \neq j\}$,

$$A(\beta,\rho) = \bigcup_{b \in \Gamma_\delta(\rho^{\alpha_d})} A(\beta,b,\rho),$$

$$A(\beta,b,\rho) = \{v \in [0,1) : \exists j \in \mathbb{Z}, \mid 2(\beta,b) + \mid b \mid^2 + \mid (j+v)\delta \mid^2 \mid < 4d_\delta \rho^{\alpha_d}\},$$

and $S_4(\beta,j,v,\rho)$ is an asymptotically full subset of F_δ:

$$\mu(S_4(\beta,j,v,\rho)) = \mu(F_\delta)(1 + O(\rho^{-\alpha})). \tag{28}$$

In this paper to obtain the asymptotic formulas, which is suitable for the constructively determination of the spectral invariants, we put an additional conditions on β, namely we suppose that

$$\beta \notin (\bigcup_{b \in \Gamma_\delta(p\rho^\alpha)} V_b^\delta(\rho^a))), \tag{29}$$

where a is defined in (9). By definition of $V_b^\delta(\rho^a)$ the relation (29) yields

$$\mid\mid \beta \mid^2 - \mid \beta + \beta_1 \mid^2 \mid \geq \rho^a, \ \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{30}$$

Using the inequalities $\mid \beta_1 \mid < p\rho^\alpha$, $\mid \tau \mid < d_\delta$, $a > 2\alpha$, we obtain

$$\mid\mid \beta + \tau \mid^2 - \mid \beta + \beta_1 + \tau \mid^2 \mid > \frac{8}{9}\rho^a, \ \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{31}$$

Now we prove (8) by using (23), (31), and the following relation called the binding formula for $L_t(q)$ and $L_t(q^\delta)$:

$$(\Lambda_N(t) - \lambda_{j,\beta}) b(N,j,\beta) = (\Psi_{N,t}(x), (q(x) - q^\delta(x))\Phi_{j,\beta}(x)), \tag{32}$$

where $b(N,j,\beta) = (\Psi_{N,t}(x), \Phi_{j,\beta}(x))$, which can be obtained from

$$L_t(q)\Psi_{n,t}(x) = \Lambda_n(t)\Psi_{n,t}(x)$$

by multiplying by $\Phi_{j,\beta}(x)$ and using $L_t(q^\delta)\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x)$. In [4], using (21), we proved that (see (3.22) and (3.23) of [4]) if $\mid j\delta \mid < r$, $\mid \beta \mid > \frac{1}{2}\rho$, where $r \geq r_1$ and $r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2 \mid \delta \mid$, then the following decomposition

$$(q(x) - q^\delta(x))\Phi_{j,\beta}(x) = \sum_{(j_1,\beta_1) \in Q(\rho^\alpha, 9r)} A(j,\beta,j+j_1,\beta+\beta_1)\Phi_{j+j_1,\beta+\beta_1}(x) + O(\rho^{-p\alpha}) \tag{33}$$

of $(q(x) - q^\delta(x))\Phi_{j,\beta}(x)$ by eigenfunction of $L_t(q^\delta)$ holds, where

$Q(\rho^{\alpha},9r)=\{(j,\beta):\mid j\delta\mid<9r,\ 0<\mid\beta\mid<\rho^{\alpha}\}$ and

$$\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\mid A(j,\beta,j+j_1,\beta+\beta_1)\mid<c_2. \tag{34}$$

Using this decomposition in (32), we get

$$(\Lambda_N(t)-\lambda_{j,\beta})b(N,j,\beta)=O(\rho^{-p\alpha})$$

$$+\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}A(j,\beta,j+j_1,\beta+\beta_1)b(N,j+j_1,\beta+\beta_1). \tag{35}$$

Remark 1 *If $\mid j^{'}\delta\mid<r$, $\mid\beta^{'}\mid>\frac{1}{2}\rho$ and $\mid\Lambda_N-\lambda_{j^{'},\beta^{'}}\mid>c(\rho)$, then by (35) we have*

$$b(N,j^{'},\beta^{'})=\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\frac{A(j^{'},\beta^{'},j^{'}+j_1,\beta^{'}+\beta_1)b(N,j^{'}+j_1,\beta^{'}+\beta_1)}{\Lambda_N-\lambda_{j^{'},\beta^{'}}}+O(\frac{1}{\rho^{p\alpha}c(\rho)}).$$

If $j\in S_1(\rho)$ then $\mid j\delta\mid<r_1=O(\rho^{\alpha_1})$ and in (35) instead of r we take r_1.

Theorem 1 *If (23) and (29) hold then the eigenvalue $\Lambda_{N(j,\beta)}(t)\equiv\Lambda_{j,\beta}(v,\tau)$ defined in (24) satisfies (8).*

Proof. Since $b(N,j,\beta)=1+O(\rho^{-\alpha_2}\ln\rho)$ (see (25)), where $N=N(j,\beta)$, we need to prove that the right-hand side of (35) is $O(\rho^{-a})$. First we show that

$$b(N,j+j_1,\beta+\beta_1)=O(\rho^{-a}) \tag{36}$$

for $\beta_1\in\Gamma_\delta(p\rho^{\alpha})$, $j=o(\rho^{\frac{a}{2}})$, $j_1=o(\rho^{\frac{a}{2}})$. For this we prove the inequality

$$\mid\Lambda_N(t)-\lambda_{j+j_1,\beta+\beta_1}\mid>\frac{1}{2}\rho^{a},\ \forall\beta_1\in\Gamma_\delta(p\rho^{\alpha}),\ \forall j=o(\rho^{\frac{a}{2}}),\forall j_1=o(\rho^{\frac{a}{2}}) \tag{37}$$

and use the formula

$$b(N,j+j_1,\beta_1+\beta)=\frac{(\Psi_{N,t}(x),(q(x)-q^{\delta}(x))\Phi_{j+j_1,\beta_1+\beta}(x))}{\Lambda_N-\lambda_{j+j_1,\beta_1+\beta}} \tag{38}$$

which can be obtained from (32) by replacing the indices j,β with $j+j_1,\beta+\beta_1$. By (24) the inequality (37) holds if

$$\mid\mu_j(v)+\mid\beta+\tau\mid^2-\mu_{j+j_1}(v)-\mid\beta+\beta_1+\tau\mid^2\mid>\frac{5}{9}\rho^{a}.$$

This inequality is consequence of inequality (31) and the equalities
$j=o(\rho^{\frac{a}{2}})$, $j+j_1=o(\rho^{\frac{a}{2}})$ (see the conditions on j, j_1 in (36), (37)),

$$\mu_n(v)=\mid(n+v)\delta\mid^2+O(\frac{1}{n}). \tag{39}$$

Thus (36) is proved. Using (36), the definition of $Q(\rho^{\alpha}, 9r_1)$, and the relations $r_1 = O(\rho^{\alpha_1})$ (see Remark 1), $\alpha_1 < \frac{a}{2}$ (see (7), (9)) we obtain that all multiplicands $b(N, j + j_1, \beta + \beta_1)$ in the right-hand side of (35), in the case $r = r_1$, is $O(\rho^{-a})$. Therefore (34) implies that the right-hand side of (35) is $O(\rho^{-a})$ ∎

To prove the asymptotic formula (9), we iterate (35) , in the case $r = r_1$, as follows. If $\mid j\delta \mid < r_1$, then the summation in (35) is taken under condition

$(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r_1)$ (see Remark 1). By the definition of $Q(\rho^{\alpha}, 9r_1)$ we have $\mid j_1\delta \mid < 9r_1$. Hence $\mid (j + j_1)\delta \mid < r_2$, where $r_2 = 10r_1$. Therefore, using (37) and Remark 1, we get

$$b(N, j + j_1, \beta_1 + \beta) = \sum_{(j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)} \frac{A(j(1), \beta(1), j(2), \beta(2))b(N, j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}},$$

where $j(k) = j+j_1+j_2+...+j_k$, $\beta(k) = \beta+\beta_1+\beta_2+...+\beta_k$ for $k = 0, 1, 2, ...$Using this in (35) we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha})+ \tag{40}$$

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{A(j, \beta, j(1), \beta(1))A(j(1), \beta(1), j(2), \beta(2))b(N, j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}.$$

To prove (9) we use this formula and the following lemma.

Lemma 1 *Suppose (23) and (29) hold. If* $j^{'} \neq j$, $\mid j^{'}\delta \mid < r$, *where* $r = O(\rho^{\frac{1}{2}\alpha_2})$, $r \geq r_1$, *and* $r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2 \mid \delta \mid$ *then*

$$b(N(j, \beta), j^{'}, \beta) = O(\rho^{-2a}r^2 \ln \rho).$$

Remark 2 *If (23) holds ,then there exists unique index* $N(j, \beta, v, \tau)$ *, depending on* j, β, v, τ, *for which the eigenvalue*$\Lambda_N(t)$ *satisfies (8). Instead of* $N(j, \beta, v, \tau)$ *we write* $N(j, \beta)$ *(or* N*) if* v, τ *(or* j, β, v, τ*) are unambiguous.*

Theorem 2 *If (23) and (29) hold, then* $\Lambda_{j,\beta}(v, \tau)$ *satisfies (9).*

Proof. We prove this by using (40). To estimate the summation in the right side of (40) we divide the terms in this summation into three group. First, second, and third group terms are the terms with multiplicands $b(N, j, \beta)$, $b(N, j(2), \beta)$ with $j(2) \neq j$, and $b(N, j(2), \beta(2))$ with $\beta(2) \neq \beta$ respectively. The sum of the first group terms is $C_1(\Lambda_N)b(N, j, \beta)$, where

$$C_1(\Lambda_N) = \sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)} \frac{A(j, \beta, j + j_1,\beta + \beta_1)A(j + j_1, \beta + \beta_1, j, \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \tag{41}$$

The sum of the second group terms is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{A(j, \beta, j + j_1,\beta + \beta_1)A(j + j_1, \beta + \beta_1, j(2), \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}b(N, j(2), \beta),$$

where $j(2) \neq j$. Since $r_2 = 10r_1 = O(\rho^{\alpha_1})$ (see Remark 1) the conditions on j, j_1, j_2 and Lemma 1 imply that $j(2) = O(\rho^{\alpha_1})$ and $b(N, j(2), \beta) = O(\rho^{-2a+2\alpha_1} \ln \rho)$. Using this, (34) and (37) we obtain that the sum of the second group terms is $O(\rho^{-3a+2\alpha_1} \ln \rho)$. The sum of the third group terms is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)}} \frac{A(j,\beta,j(1),\beta(1))A(j(1),\beta(1),j(2),\beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}} b(N,j(2),\beta(2)), \quad (42)$$

where $\beta(2) \neq \beta$. Using (37) and Remark 1 we get

$$b(N,j(2),\beta(2)) = \sum_{(j_3,\beta_3)\in Q(\rho^\alpha,9r_3)} \frac{A(j(2),\beta(2),j(3),\beta(3))b(N,j(3),\beta(3))}{\Lambda_N - \lambda_{j(2),\beta(2)}} + O(\rho^{-p\alpha}),$$

where $r_3 = 10r_2$. Substituting it into (42) and isolating the terms with multiplicands $b(N, j, \beta)$ we see that the sum of the third group terms is

$$C_2(\Lambda_N) b(N,j,\beta) + C_3(\Lambda_N) + O(\rho^{-p\alpha}),$$

where

$$C_2(\Lambda_N) = \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1),\\(j_2,\beta_2)\in Q(\rho^\alpha,90r_1)}} \frac{A(j,\beta,j(1),\beta(1))A(j(1),\beta(1),j(2),\beta(2))A(j(2),\beta(2),j,\beta)}{(\Lambda_N - \lambda_{j+j_1,\beta+\beta_1})(\Lambda_N - \lambda_{j(2),\beta(2)})}, \quad (43)$$

$$C_3(\Lambda_N) = \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\(j_2,\beta_2)\in Q(\rho^\alpha,9r_2),\\(j_3,\beta_3)\in Q(\rho^\alpha,9r_3)}} \frac{\left(\prod_{k=1,2,3} A(j(k-1),\beta(k-1),j(k),\beta(k))\right) b(N,j(3),\beta(3))}{(\Lambda_N - \lambda_{j(1),\beta(1)})(\Lambda_N - \lambda_{j(2),\beta(2)})},$$

and $(j(3), \beta(3)) \neq (j, \beta)$. By (36) and Lemma 1 $b(N, j(3), \beta(3)) = O(\rho^{-a})$ for $(j(3), \beta(3)) \neq (j, \beta)$. Using this, (34), and taking into account that

$$| \Lambda_N(t) - \lambda_{j(1),\beta(1)} | > \frac{1}{3}\rho^a, \quad | \Lambda_N(t) - \lambda_{j(2),\beta(2)} | > \frac{1}{3}\rho^a$$

for $\beta(1) \neq \beta$, $\beta(2) \neq \beta$ (see (37)), we obtain $C_3(\Lambda_N) = O(\rho^{-3a})$. The estimations of the first, second and third groups terms imply that the formula (40) can be written in the form

$$(\Lambda_N - \lambda_{j,\beta}) b(N,j,\beta) = (C_1(\Lambda_N) + C_2(\Lambda_N)) b(N,j,\beta) + O(\rho^{-3a+2\alpha_1} \ln \rho), \quad (44)$$

where $N = N(j, \beta, v, \tau)$, $\Lambda_N = \Lambda_{j,\beta}(v, \tau)$. Therefore dividing both part of (44) by $b(N, j, \beta)$, where $b(N, j, \beta) = 1 + o(1)$ (see (25)), we get

$$\Lambda_{j,\beta} = \lambda_{j,\beta} + C_1(\Lambda_{j,\beta})) + C_2(\Lambda_{j,\beta})) + O(\rho^{-3a+2\alpha_1} \ln \rho). \quad (45)$$

The calculations in Appendix C and in Appendix B show that

$$C_1(\Lambda_{j,\beta}(v,\tau))=\frac{1}{4}\int_F |f_{\delta,\beta+\tau}(x)|^2\,|\varphi_{j,v}|^2\,dx+O(\rho^{-3a+2\alpha_1}), \tag{46}$$

$$C_2(\Lambda_{j,\beta}(v,\tau))=O(\rho^{-3a+2\alpha_1}). \tag{47}$$

Therefore (9) follows from (45) ■

Theorem 3 *If (23) and (29) hold then the eigenfunction $\Psi_{j,\beta}(x)$ corresponding to $\Lambda_{j,\beta}(v,\tau)$ satisfies (10).*

Proof. To prove (10) we need to show that

$$\sum_{(j',\beta'):(j',\beta')\neq(j,\beta)} |\ b(N(j,\beta),j^{'},\beta^{'})\ |^2=O(\rho^{-2a}). \tag{48}$$

In [4] (see (6.36) of [4]) we proved that

$$\sum_{(j',\beta')\in S^c(k-1)} |\ b(N,j^{'},\beta^{'})\ |^2=O(\rho^{-2k\alpha_2}(\ln\rho)^2), \tag{49}$$

where $S^c(n)=K_0\backslash S(n)$, $K_0=\{(j^{'},\beta^{'}):j^{'}\in\mathbb{Z},\beta^{'}\in\Gamma_\delta,(j^{'},\beta^{'})\neq(j,\beta)\}$, $S(n)=\{(j^{'},\beta^{'})\in K_0:|\ \beta-\beta^{'}\ |\leq n\rho^{\alpha}\ ,|\ j^{'}\delta\ |<10^n h\}$, $h=O(\rho^{\frac{1}{2}\alpha_2})$ and k can be chosen such that $k\alpha_2>a$, $k<p$. Therefore it is enough to prove that

$$\sum_{(j',\beta')\in S(k-1)} |\ b(N,j^{'},\beta^{'})\ |^2=O(\rho^{-2a}). \tag{50}$$

Using (37), (38), definition of $S(k-1)$ and Bessel inequality for the basis $\{\Phi_{j',\beta'}(x):j^{'}\in\mathbb{Z},\ \beta^{'}\in\Gamma_\delta\}$ we have

$$\sum_{(j',\beta'):(j',\beta')\in S(k-1),\beta'\neq\beta} |\ b(N,j^{'},\beta^{'})\ |^2=$$

$$\sum_{(j',\beta')}\frac{|\ (\Psi_N(x)(q(x)-Q(s)),\Phi_{j^{'},\beta'}(x))\ |^2}{|\ \Lambda_N-\lambda_{j',\beta'}\ |^2}=O(\rho^{-2a}). \tag{51}$$

In the case $\beta^{'}=\beta$, $j^{'}\neq j$ using Lemma 1 (we can use it since $|\ j^{'}\delta\ |=O(\rho^{\frac{1}{2}\alpha_2})$ for $(j^{'},\beta^{'})\in S(k-1)$), we obtain

$$\sum_{(j',\beta)\in S(k-1),j'\neq j} |\ b(N,j^{'},\beta)\ |^2=O(\rho^{-4a+2\alpha_2}(\ln\rho)^2)K, \tag{52}$$

where K is the number of $j^{'}$ satisfying $(j^{'},\beta)\in S(k-1)$. It is clear that $K=O(\rho^{\frac{1}{2}\alpha_2})$. Since $\alpha_2<\frac{a}{2}$ (see (7) and (9)), the right side of (52) is $O(\rho^{-2a})$. Therefore (52) and (51) give (50) ■

Now we estimate the derivatives of $\Lambda_N(t)$ by using the following lemma.

Lemma 2 *Let $\Lambda_N(\beta+\tau+(j+v)\delta)$, be a simple eigenvalue of L_t satisfying*

$$\mid \Lambda_N(\beta+\tau+(j+v)\delta)-\lambda_{j,\beta}(v,\tau)\mid < 1, \tag{53}$$

where j,β, satisfy (23), and $\beta+\tau+(j+v)\delta-t\in\Gamma$. Then

$$\mid \beta+\tau \mid \frac{\partial \Lambda_N(t)}{\partial h}=\sum_{j^{'}\in\mathbb{Z},\ \beta^{'}\in\Gamma_\delta}(\beta+\tau,\beta^{'}+\tau)\mid b(N,j^{'},\beta^{'})\mid^2, \tag{54}$$

where $\frac{\partial \Lambda_N(t)}{\partial h}$ is the derivative of $\Lambda_N(t)$ in the direction $h=\frac{\beta+\tau}{|\beta+\tau|}$. Moreover

$$\mid b(N,j^{'},\beta^{'})\mid\leq \frac{c_3}{(\mid\beta^{'}+\tau\mid^2+\mid(j^{'}+v)\delta\mid^2)\mid\beta^{'}+\tau\mid^{2d+6}} \tag{55}$$

for all $\beta^{'}$ satisfying $\mid\beta^{'}+\tau\mid\geq 4\rho$ and for all $j^{'}\in\mathbb{Z}$.

Theorem 4 *If (23) and (29) hold, then (11) holds too.*

Proof. It follows from (55), (48), (10) that

$$\sum_{j^{'}\in\mathbb{Z},|\beta^{'}+\tau|\geq 4\rho}(\beta+\tau,\beta^{'}+\tau)\mid b(N,j^{'},\beta^{'})\mid^2=O(\rho^{2-2a}),$$

$$\sum_{j^{'}\in\mathbb{Z},|\beta^{'}+\tau|<4\rho,(j^{'},\beta^{'})\neq(j,\beta)}(\beta+\tau,\beta^{'}+\tau)\mid b(N,j^{'},\beta^{'})\mid^2=O(\rho^{2-2a}),$$

$$(\beta+\tau,\beta+\tau)\mid b(N,j,\beta)\mid^2=\mid\beta+\tau\mid^2+O(\rho^{2-2a}),$$

where $N=N(j,\beta,v,\tau)$ (see Remark 2). Therefore (11) follows from (54) ■

To prove the main results of this paper we need the following lemmas

Lemma 3 *If $\Lambda_N(\beta+\tau+(j+v)\delta)$ is a simple eigenvalue of $L_t(q)$, where $\beta+\tau+(j+v)\delta-t\in\Gamma$, satisfying (53) and $N\neq N(j,\beta)$, then*

$$\mid\beta+\tau\mid\frac{\partial\Lambda_N(t)}{\partial h}<\mid\beta+\tau\mid^2-\frac{1}{4}\rho^{2\alpha_d}.$$

The proof of this lemma is given in Section 3. Here we only note some reason of this estimations. It follows from (48) that

$$\mid b(N,j,\beta)\mid^2=1+O(\rho^{-2a}) \text{ for } N=N(j,\beta). \tag{56}$$

Since $\parallel\Phi_{j,\beta}(x)\parallel=1$, using Parseval's equality for the orthonormal basis $\{\Psi_N(x):N=1,2,...,\}$ and (56), we get

$$\mid b(N,j,\beta)\mid^2=O(\rho^{-2a}),\ \forall\ N\neq N(j,\beta). \tag{57}$$

This with the long technical estimations of the other term of the series of the right side of (54) implies the proof of the Lemma 3.

Lemma 4 *Let b be a maximal element of Γ_δ and $v \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Then there exist $\rho(v)$ such that for $\rho \geq \rho(v)$ there exists $\beta \in S_2(\rho)$ satisfying the relation $v \notin A(\beta,\rho)$ and the inequalities*

$$\frac{1}{3}|\rho|^a <| (\beta+\tau,b) |< 3|\rho|^a, \tag{58}$$

$$| (\beta+\tau,\gamma) |> \frac{1}{3}|\rho|^a, \ \forall\gamma \in S(\delta,b)\backslash\delta\mathbb{R}, \tag{59}$$

$$| (\beta+\tau,\gamma) |> \frac{1}{3}|\rho|^{a+2\alpha}, \ \forall\gamma \notin S(\delta,b), \ \ | \gamma |< |\rho|^\alpha, \tag{60}$$

$$\int_F |f_{\delta,\beta+\tau}(x)|^2 \, |\varphi_{n,v}|^2 \, dx < c_4\rho^{-2a} \tag{61}$$

for $\tau \in F_\delta$, where S_2, $A(\beta,\rho)$, $f_{\delta,\beta+\tau}$, $S(\delta,b)$ are defined in (26), (27), (9), (13).

Theorem 5 *Suppose $q(x) \in W_2^s(F)$ and the band functions are known. Then the spectral invariants $\mu_n(v)$ for $n \in \mathbb{Z}$, $v \in [0,1)$ and (12), (15), (16), (17), (20) can be constructively determined.*

Proof. Take any $j \in \mathbb{Z}$ and $v \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. In [4] (see Lemma 3.7 of [4]) we proved that

$$(\varepsilon(\rho),\frac{1}{2}-\varepsilon(\rho)) \cup (\frac{1}{2}+\varepsilon(\rho), 1-\varepsilon(\rho)) \subset W(\rho),$$

where $W(\rho)$ is defined in (27) and $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. Therefore $v \in W(\rho)$ for $\rho \gg 1$. On the other hand by Lemma 4 there exists $\beta \in S_2(\rho)$ such that $v \notin A(\beta,\rho)$ and (58)-(61) holds. Then $v \in S_3(\beta,\rho)$ (see (27)). Thus j,β,v satisfy (23) and β satisfies (58)-(61) for $\rho \gg 1$. Replacing ρ by $\rho_k \equiv 3^k\rho$ for

$k = 1,2,...$, in the same way we obtain the sequence β_1,β_2,..., such that $\beta_k \in S_2(\rho_k)$, $v \in S_3(\beta_k,\rho_k)$ and the inequalities obtained from (58)-(61) by replacing β,ρ with β_k,ρ_k holds. Now take τ from F_δ and consider the band functions $\Lambda_N(\beta_k+\tau+(j+v)\delta)$ for $N = 1,2,....$ Let A_k be the set of all $\tau \in F_\delta$ for which there is N satisfying the conditions:

$$| \Lambda_N(\beta_k+\tau+(j+v)\delta) - | \beta_k+\tau |^2 - | (j+v)\delta |^2 |< 1, \tag{62}$$

$$\Lambda_N(\beta_k+\tau+(j+v)\delta) \text{ is a simple eigenvalue,} \tag{63}$$

$$|| \beta_k+\tau | \frac{\partial\Lambda_N(\beta_k+\tau+(j+v)\delta)}{\partial h} - | \beta_k+\tau |^2 |< \rho_k^{2-2a+\alpha}, \tag{64}$$

where $h = \frac{\beta_k+\tau}{|\beta_k+\tau|}$. For $\tau \in A_k$ take one of eigenvalues $\Lambda_N(\beta_k+\tau+(j+v)\delta)$ satisfying (62)-(64) and calculate the integral

$$J(A_k) = \frac{1}{\mu(F_\delta)} \int_{A_k} (\Lambda_N(\beta_k+\tau+(j+v)\delta) - | \beta_k+\tau |^2) d\tau.$$

We write these integral as sum of $J(S_4(\beta_k,j,v,\rho_k))$ and $J(A_k\backslash S_4(\beta_k,j,v,\rho_k))$, where $S_4(\beta_k,j,v,\rho_k)$ is defined in (28). Note that if $\tau \in S_4(\beta,j,v,\rho)$, where

$j \in S_1(\rho)$, $\beta \in S_2(\rho)$, $v \in S_3(\beta,\rho)$ (see (23)) then the formulas (24), (8), (9), (10), (11) hold. Here for brevity of notations instead of $S_4(\beta_k, j, v, \rho_k)$ we will write S_4. If $\tau \in S_4$ then by (24) and Theorem 4 the eigenvalue $\Lambda_{j,\beta_k}(v,\tau)$ satisfies the conditions (62)-(64) and by Lemma 3 the other eigenvalues does not satisfy these conditions. Hence in the integral $J(S_4)$ instead of

$\Lambda_N(\beta_k + \tau + (j+v)\delta)$ we must take $\Lambda_{j,\beta_k}(v,\tau))$. Therefore using (8), (28), the inclusion $A_k \subset F_\delta$, and (62), we get

$$J(S_4) = \mu_j(v) + O(\rho_k^{-a}),\ \mu(A_k \backslash S_4) = O(\rho_k^{-a}),\ J(A_k \backslash S_4) = O(\rho^{-a}).$$

These equalities imply that

$$J(A_k) = \mu_j(v) + O(\rho_k^{-a}).$$

Now tending k to ∞ we find the eigenvalue $\mu_j(v)$ for $j \in \mathbb{Z}$ and $v \in (0,\frac{1}{2}) \cup (\frac{1}{2}, 1)$. Since $\mu_j(0)$ and $\mu_j(\frac{1}{2})$ are the end points of the interval $\{\mu_j(v) : v \in (0,\frac{1}{2})\}$ the invariants $\mu_j(v)$ are constructively determined for all $j \in \mathbb{Z}$, $v \in [0,1)$. In the Appendix D we constructively determine (17) from the asymptotic formulas for $\mu_j(v)$.

Let B_k be the set of $\tau \in F_\delta$ for which there is N satisfying (63), (64), and

$$\mid \Lambda_N(\beta_k + \tau + (j+v)\delta) - \mid \beta_k + \tau \mid^2 - \mu_j(v) \mid < \rho_k^{-2a+\frac{\alpha}{2}}, \tag{65}$$

For $\tau \in B_k$ take one of the eigenvalues $\Lambda_N(\beta_k + \tau + (j+v)\delta)$ satisfying (63)-(65) and estimate the integral

$$J^{'}(B_k) = \frac{\mid (\beta_k + \tau, b) \mid^2}{\mu(F_\delta)|b|^4} \int_{B_k} (\Lambda_N(\beta_k + \tau + (j+v)\delta) - \mid \beta_k + \tau \mid^2 - \mu_j(v)) d\tau.$$

We write these integral as sum of $J^{'}(S_4)$ and $J^{'}(B_k \backslash S_4)$. If $\tau \in S_4$ then by Theorem 4 and Lemma 3 only $\Lambda_{j,\beta_k}(v,\tau))$ satisfies (63)-(65). Hence in the integral $J^{'}(S_4)$ instead of $\Lambda_N(\beta_k + \tau + (j+v)\delta)$ we must take $\Lambda_{j,\beta_k}(v,\tau))$. Therefore using (9), (58) we get

$$J^{'}(S_4) = \frac{\mid (\beta_k + \tau, b) \mid^2}{\mu(F_\delta)|b|^4} \int_{S_4} \int_F \mid f_{\delta,\beta_k+\tau}(x) \varphi_{j,v} \mid^2 dx d\tau + O(\rho_k^{2\alpha_1 - a} \ln \rho). \tag{66}$$

Moreover using (65), (58), and $\mu(B_k \backslash S_4) = O(\rho_k^{-\alpha})$ (see (28)), we obtain

$$J^{'}(B_k \backslash S_4) = O(\rho_k^{-\frac{\alpha}{2}}). \tag{67}$$

Substituting the decomposition $|\delta|^{-2}(\gamma,\delta)\delta + |b|^{-2}(\gamma, b)b$ of γ for $\gamma \in S(\delta, b)$, $\mid \gamma \mid < |\rho_k|^\alpha$ into the denominator of the fraction in $f_{\delta,\beta_k+\tau}(x)$ (for definition of this function see (9)) and using (58), (60) we have

$$\lim_{k\to\infty} |b|^{-2}(\beta_k + \tau, b) f_{\delta,\beta_k+\tau}(x) = \sum_{\gamma \in S(\delta,b)\backslash \delta\mathbb{R}} \frac{\gamma}{(\gamma, b)} q_\gamma e^{(\gamma,x)} \equiv q_{\delta,b}(x), \tag{68}$$

where $q_{\delta,b}(x)$ is defined in (13) and the convergence of the series (13) is proved in the proof of Lemma 4. This with (66) and (67) implies that

$$\lim_{k\to\infty} J^{'}(B_k) = \int_F |q_{\delta,b}(x)|^2 \, |\varphi_{j,v}|^2 \, dx \equiv J(\delta, b, j, v) \tag{69}$$

(see (12)). In (69) tending j to infinity and using (14) we get the invariant $J_0(\delta, b)$ (see (15)). Then we find the other invariants $J_1(\delta, b), J_2(\delta, b), ...,$ of (15) as follows

$J_1 = \lim_{j\to\infty}(J - J_0)j,\ J_2 = \lim_{j\to\infty}((J - J_0)j^2 - J_1 j), ...$

In the Appendix D using the asymptotic formulas for the eigenfunctions of $T_v(Q)$ we constructively determine the invariants (16), (20) from (15) and (17) ■

3 The proofs of the lemmas

The proof of Lemma 1

To prove this lemma we use the following formula obtained from (40) by replacing j and r_1 with $j^{'}$ and r respectively

$$(\Lambda_{N(j,\beta)} - \lambda_{j^{'},\beta}) b(N, j^{'}, \beta) = O(\rho^{-p\alpha}) + \tag{70}$$

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r)\\(j_2,\beta_2)\in Q(\rho^\alpha,90r)}} \frac{A(j,\beta,j^{'}(1),\beta(1))A(j^{'}(1),\beta(1),j^{'}(2),\beta(2))b(N,j^{'}(2),\beta(2))}{\Lambda_N - \lambda_{j^{'}+j_1,\beta+\beta_1}},$$

where $j^{'}(k) = j^{'} + j_1 + j_2 + ... + j_k$ for $k = 0, 1, 2, ...$By (36) we have

$$b(N, j^{'}(2), \beta(2)) = O(\rho^{-a}) \tag{71}$$

for $\beta(2) \neq \beta$. If $j^{'}(2) \neq j$, then using (8) and taking into account that $v \in S_3(\beta, \rho) \subset W(\rho)$ (see the definition of $W(\rho)$ in (27)) we obtain

$$| \Lambda_{N(j,\beta)} - \lambda_{j^{'},\beta} | > \frac{1}{\ln \rho}. \tag{72}$$

Therefore using (34), Remark 1, and (36) we see that

$$b(N, j^{'}(2), \beta) = O(\rho^{-a} \ln \rho)$$

for $j^{'}(2) \neq j$. Using this, (34), and the estimations (37), (71) we see that the sum of the terms of the right-hand side of (70) with multiplicand $b(N, j^{'}(2), \beta(2))$ for $(j^{'}(2), \beta(2)) \neq (j, \beta)$ is $O(\rho^{-2a} \ln \rho)$. It means that the formula (70) can be written in the form

$$(\Lambda_N - \lambda_{j^{'},\beta}) b(N, j^{'}, \beta) = O(\rho^{-2a} \ln \rho) + C_1(j^{'}, \Lambda_N) b(N, j, \beta), \tag{73}$$

where

$$C_1(j^{'},\Lambda_N)=\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\frac{A(j^{'},\beta,j^{'}+j_1,\beta+\beta_1)A(j^{'}+j_1,\beta+\beta_1,j,\beta)}{\Lambda_N-\lambda_{j^{'}+j_1,\beta+\beta_1}}. \tag{74}$$

By (8), (37), and (34) we have

$$\frac{1}{\Lambda_N-\lambda_{j^{'}+j_1,\beta+\beta_1}}=\frac{1}{\lambda_{j,\beta}-\lambda_{j^{'}+j_1,\beta+\beta_1}}=O(\rho^{-3a})$$

and

$$C_1(j^{'},\Lambda_N)=C_1(j^{'},\lambda_{j,\beta})+O(\rho^{-3a}), \tag{75}$$

where $C_1(j^{'},\lambda_{j,\beta})$ is obtained from $C_1(j^{'},\Lambda_N)$ by replacing Λ_N with $\lambda_{j,\beta}$ in the denominator of the fractions in (74). In Appendix A we prove that

$$C_1(j^{'},\lambda_{j,\beta})=O(\rho^{-2a}r^2) \tag{76}$$

for $\mid j^{'}\delta\mid<r$, $(j_1,\beta_1)\in Q(\rho^{\alpha},9r)$, $j\in S_1$. Therefore dividing both sides of (73) by $\Lambda_N-\lambda_{j^{'},\beta}$ and using (72), (75), (76) we get the proof of the lemma.

The proof of Lemma 2

We calculate the derivative of $\Lambda_N(t)$ by using the formula

$$\frac{\partial\Lambda_N(t)}{\partial t_j}=2t_j-2i(\frac{\partial}{\partial x_j}\Phi_{N,t}(x),\Phi_{N,t}(x)),$$

where $\Phi_{N,t}(x)=e^{-i(t,x)}\Psi_{N,t}(x)$, $t=(t_1,t_2,...,t_d)$ (see (5.12) of [4]). Then

$$\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j=1}^{d}h_j\frac{\partial\Lambda_N(t)}{\partial t_j}=2(h,t)-2i(\frac{\partial}{\partial h}\Phi_{N,t}(x),\Phi_{N,t}(x)). \tag{77}$$

To compute $\frac{\partial}{\partial h}\Phi_{N,t}(x)$ we prove that the decomposition

$$\Phi_{N,t}(x)=\sum_{j^{'}\in\mathbb{Z},\beta^{'}\in\Gamma_\delta}b(N,j^{'},\beta^{'})e^{i(\beta^{'}+\tau-t,x)}\varphi_{j^{'}}((\delta,x)) \tag{78}$$

of $\Psi_{N,t}(x)$ over basis $\{\Psi_{j,\beta}(x):j\in\mathbb{Z},\beta\in\Gamma_\delta\}$ can be differentiated term by term. Since $(\delta,h)=0$,

$$\frac{\partial}{\partial h}e^{i(\beta^{'}+\tau-t,x)}\varphi_{j^{'}}((\delta,x))=i(\beta^{'}+\tau-t,h)e^{i(\beta^{'}+\tau-t,x)}\varphi_{j^{'}}((\delta,x)),$$

we need to prove that

$$\frac{\partial}{\partial h}\Phi_{N,t}(x)=\sum_{j^{'}\in\mathbb{Z},\beta^{'}\in\Gamma_\delta}i(\beta^{'}+\tau-t,h)b(N,j^{'},\beta^{'})e^{i(\beta^{'}+\tau-t,x)}\varphi_{j^{'}}((\delta,x)). \tag{79}$$

Therefore we consider the convergence of these series by estimating the multiplicand $b(N,j^{'},\beta^{'})$. First we estimate this multiplicand for $(j^{'},\beta^{'})\in E$, where $E=\{(j^{'},\beta^{'}):\mid (j^{'}+v)\delta\mid^{2}+\mid\beta^{'}+\tau\mid^{2}\geq 9\rho^{2}\}$ by using the formula

$$b(N,j^{'},\beta^{'})=\frac{(\Psi_{N,t}(x),(q(x)-q^{\delta}(x))\Phi_{j^{'},\beta^{'}}(x))}{\Lambda_{N}-\lambda_{j^{'},\beta^{'}}} \tag{80}$$

which can be obtained from (38) by replacing the indices $j+j_{1},\beta+\beta_{1}$ with $j^{'},\beta^{'}$. It follows from (8) and (23) that

$$\mid\Lambda_{N}\mid<3\rho^{2}. \tag{81}$$

This inequality, the condition $(j^{'},\beta^{'})\in E$, definition of $\lambda_{j^{'},\beta^{'}}$ and (39) give

$$\lambda_{j^{'},\beta^{'}}-\Lambda_{N}>\frac{1}{2}(\mid (j^{'}+v)\delta\mid^{2}+\mid\beta^{'}+\tau\mid^{2})>\rho^{2} \tag{82}$$

for $(j^{'},\beta^{'})\in E$. Therefore (80) implies that

$$\mid b(N,j^{'},\beta^{'})\mid\leq\frac{c_{5}}{\mid (j^{'}+v)\delta\mid^{2}+\mid\beta^{'}+\tau\mid^{2}},\ \forall(j^{'},\beta^{'})\in E. \tag{83}$$

Now we obtain the high order estimation for $b(N,j^{'},\beta^{'})$ when $\mid\beta^{'}+\tau\mid\geq 4\rho$. In this case to estimate $b(N,j^{'},\beta^{'})$ we use the iterations of the formula in Remark 1. To iterate this formula we use the following obvious relations

$$\mid\beta^{'}+\tau-\beta_{1}-\beta_{2}-...-\beta_{k}\mid^{2}>\frac{3}{4}\mid\beta^{'}+\tau\mid^{2}$$

for $k=1,2,...,d+3$, where $\mid\beta_{i}\mid<\rho^{\alpha}$ for $i=0,1,...,k$. This and (81) give

$$\lambda_{j^{'}(k),\beta^{'}(k)}-\Lambda_{N}>\frac{1}{5}\mid\beta^{'}+\tau\mid^{2},\forall\mid\beta^{'}+\tau\mid\geq 4\rho, \tag{84}$$

where $\beta^{'}(k)=\beta^{'}+\beta_{1}+\beta_{2}+...+\beta_{k}$. Moreover if $\mid j^{'}\delta\mid<c$, where c is a positive number, then $(j_{k},\beta_{k})\in Q(\rho^{\alpha},10^{k-1}9c)$. These conditions on $j^{'}$ and j_{1} imply that $\mid j^{'}(1)\delta\mid<10c$. Therefore in the formula in Remark 1 replacing $j^{'},\beta^{'},r$ by $j^{'}(1),\beta^{'}(1),10c$, we get

$$b(N,j^{'}(1),\beta^{'}(1))=O(\rho^{-p\alpha})+$$
$$\sum_{(j_{2},\beta_{2})\in Q(\rho^{\alpha},90c)}\frac{A(j^{'}(1),\beta^{'}(1),j^{'}(2),\beta^{'}(2))b(N,j^{'}(2),\beta^{'}(2))}{\Lambda_{N}-\lambda_{j^{'}(1),\beta^{'}(1)}}.$$

In the same way we obtain

$$b(N,j^{'}(k),\beta^{'}(k))=O(\rho^{-p\alpha})+ \tag{85}$$
$$\sum_{(j_{k+1},\beta_{k+1})\in Q(\rho^{\alpha},(10^{k})9c)}\frac{A(j^{'}(k),\beta^{'}(k),j^{'}(k+1),\beta^{'}(k+1))b(N,j^{'}(k+1),\beta^{'}(k+1))}{\Lambda_{N}-\lambda_{j^{'}(k),\beta^{'}(k)}}$$

for $k=1,2,....$ In the formula in Remark 1 for $r=c$ using this formula for $k=1,2,...d+3$ successively, we get

$$b(N,j^{'},\beta^{'})=\sum(\prod_{i=0}^{d+3}\frac{A(j^{'}(i),\beta^{'}(i),j^{'}(i+1),\beta^{'}(i+1))}{(\Lambda_N-\lambda_{j^{'}(i),\beta^{'}(i)})})b(N,j^{'}(d+4),\beta^{'}(d+4)), \tag{86}$$

where sum is taken under conditions $(j_1,\beta_1)\in Q(\rho^{\alpha},9c),(j_2,\beta_2)\in Q(\rho^{\alpha},90c),...,$ $(j_{d+4},\beta_{d+4})\in Q(\rho^{\alpha},(10^{d+3})9c)$. Now using (34), (82), and (84), we obtain the proof of (55). It follows from (83) and (55) that the series in (78) can be term by term differentiated and (79) holds. Substituting (79) into (77) and using the Parseval equality by direct calculation we obtain the proof of the lemma.

The proof of Lemma 3

By Lemma 2 we have

$$\mid\beta+\tau\mid\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j^{'}\in\mathbb{Z},\beta^{'}\in\Gamma_\delta}(\beta+\tau,\beta^{'}+\tau)\mid b(N,j^{'},\beta^{'})\mid^2=\sum_{i=1}^{7}C_i, \tag{87}$$

where

$$C_i=\sum_{\beta^{'}\in A_i}\sum_{j^{'}\in\mathbb{Z}}(\beta+\tau,\beta^{'}+\tau)\mid b(N,j^{'},\beta^{'})\mid^2 \tag{88}$$

and A_i is defined as follows

$A_1=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\notin R_\delta(4\rho)\}$,

$A_2=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in R_\delta(4\rho)\backslash R_\delta(H+\frac{1}{9}\rho^{a-1})\}$,

$A_3=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in(R_\delta(H+\frac{1}{9}\rho^{a-1})\backslash R_\delta(H+\rho^{\alpha_d-1})),\mid\beta-\beta^{'}\mid\geq\rho^{a-2\alpha}\}$,

$A_4=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in(R_\delta(H+\frac{1}{9}\rho^{a-1})\backslash R_\delta(H+\rho^{\alpha_d-1})),\mid\beta-\beta^{'}\mid<\rho^{a-2\alpha}\}$,

$A_5=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in(R_\delta(H+\rho^{\alpha_d-1})\backslash R_\delta(H-\rho^{2\alpha_d-1})),\mid\beta-\beta^{'}\mid\geq\rho^{\alpha_d}\}$,

$A_6=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in(R_\delta(H+\rho^{\alpha_d-1})\backslash R_\delta(H-\rho^{2\alpha_d-1})),\mid\beta-\beta^{'}\mid<\rho^{\alpha_d}\}$

$A_7=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in R_\delta(H-\rho^{2\alpha_d-1})\}$, where $H=\mid\beta+\tau\mid$, $\beta\in S_2(\rho)$, and hence by definition of $S_2(\rho)$ (see (23)) H satisfies the inequalities

$$\frac{1}{2}\rho<H<\frac{3}{2}\rho. \tag{89}$$

First we prove that

$$C_i=O(\rho^{2-2a}),\ \forall i=1,2,4,6. \tag{90}$$

It follows from (55) that (90) holds for $i=1$. To prove (90) for $i=2$ we use (80) and show that

$$\lambda_{j^{'},\beta^{'}}-\Lambda_N(t)>c_6\rho^a. \tag{91}$$

First let us prove (91). Since (53) holds, it follows from (8), (39) and the relations $j\in S_1(\rho)$ (see (23) and definition of $S_1(\rho)$) that

$$\Lambda_N=H^2+O(\rho^{2\alpha_1}). \tag{92}$$

If $\beta^{'} \in A_2$ then using (89), definition of $\lambda_{j^{'},\beta^{'}}$,, and (39) we have

$$\lambda_{j^{'},\beta^{'}} > H^2 + c_7\rho^a. \tag{93}$$

This, (92), and the inequality $a > 2\alpha_1$ imply (91). Now using (91), (80), the inequalities $\mid \beta + \tau \mid < \frac{3}{2}\rho$ (see (89)), $\mid \beta^{'} + \tau \mid < 4\rho$ and Bessel inequality we obtain the prove of (90) for $i = 2$.

To prove (90) for $i = 4$ we use the inequality $C_4 < c_8\rho^2(C_{4,1} + C_{4,2})$, where

$$C_{4,1} = \sum_{\beta^{'} \in A_4} (\sum_{j^{'}:|j^{'}\delta| \geq \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N, j^{'}, \beta^{'}) \mid^2),$$

$$C_{4,2} = \sum_{\beta^{'} \in A_4} (\sum_{j^{'}:|j^{'}\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N, j^{'}, \beta^{'}) \mid^2)$$

and prove that

$$C_{4,i} = O(\rho^{-2a}), \forall i = 1, 2. \tag{94}$$

It is clear that if $\beta^{'} \in A_4$ and $\mid j^{'}\delta \mid \geq \frac{1}{30}\rho^{\frac{a}{2}}$ then (93) holds. Therefore repeating the prove of (90) for $i = 2$ we get the proof of (94) for $i = 1$.

Now we prove (94) for $i = 2$. It follows from (80) that

$$C_{4,2} = \sum_{\beta^{'} \in A_4} (\sum_{j^{'}:|j^{'}\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{\mid (\Psi_N(x), (q(x) - Q(s))\Phi_{j^{'},\beta^{'}}(x)) \mid^2}{\mid \Lambda_N(t) - \lambda_{j^{'},\beta^{'}} \mid^2}. \tag{95}$$

Since $\alpha_d > 2\alpha_1$ it follows from (92) that the inequality $\lambda_{j^{'},\beta^{'}} - \Lambda_N(t) > c_9\rho^{\alpha_d}$ holds for $\beta^{'} \in A_4$, $\mid j^{'}\delta \mid < \rho^{\frac{a}{2}}$. Therefore using (39) we obtain

$$\sum_{j^{'}:|j^{'}\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{1}{\mid \Lambda_N(t) - \lambda_{j^{'},\beta^{'}} \mid^2} < c_{10}, \ \forall \beta^{'} \in A_4, \tag{96}$$

where c_{10} does not depend on $\beta^{'}$. Using this in (95) and denoting

$$\mid (\Psi_N, (q(x) - Q(s))\Phi_{n(\beta^{'}),\beta^{'}}(x)) \mid = \max_{j^{'}:|j^{'}\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \mid (\Psi_N, (q(x) - Q(s))\Phi_{j^{'},\beta^{'}}(x)) \mid$$

(if max is gotten for several index $n(\beta^{'})$, then we take one of them), we get

$$C_{4,2} < c_{11} \sum_{\beta^{'} \in A_4} \mid (\Psi_N(x), (q(x) - Q(s))\Phi_{n(\beta^{'}),\beta^{'}}(x)) \mid^2 .$$

Now using (33), (34) and then (80) we obtain

$$C_{4,2} < c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta^{'} \in A_4} \mid b(N, n(\beta^{'}) + j_1(\beta^{'}), \beta^{'} + \beta_1(\beta^{'})) \mid^2 \tag{97}$$

$$= c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta^{'} \in A_4} \frac{\mid (\Psi_N, (q(x) - Q(s))\Phi_{n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'})}(x)) \mid^2}{\mid \Lambda_N - \lambda_{n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'})} \mid^2},$$

where

$$| b(N, n(\beta^{'})+j_1(\beta^{'}), \beta^{'}+\beta_1(\beta^{'})) |= \max_{(j_1,\beta_1)\in Q(\rho^{\alpha}, 9\frac{1}{30}\rho^{\frac{a}{2}})} | b(N, n(\beta^{'})+j_1, \beta^{'}+\beta_1) | .$$

To estimate $C_{4,2}$ let us prove that

$$| \Lambda_N - \lambda_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')} |> \frac{1}{8}\rho^a. \tag{98}$$

The inclusion $(j_1, \beta_1) \in Q(\rho^{\alpha}, 9\frac{1}{30}\rho^{\frac{a}{2}})$ and the condition $| j^{'}\delta |< \frac{1}{30}\rho^{\frac{a}{2}}$ imply that $| n(\beta^{'})\delta + j_1(\beta^{'})\delta |< \frac{1}{3}\rho^{\frac{a}{2}}$ and by (39)

$$| \mu_{n(\beta')+j_1(\beta')} |< \frac{1}{8}\rho^a.$$

Therefore, by (92), to prove (98) it is enough to show that

$$| H^2- | \beta^{'} + \beta_1 + \tau |^2|> \frac{3}{8}\rho^a, \forall\beta^{'} \in A_4, \beta_1 \in \Gamma_\delta(p\rho^{\alpha}). \tag{99}$$

Since $|| \beta^{'} + \tau |^2 -H^2 |< \frac{1}{2}\rho^a$ (see definition of A_4 and use (89)) we need to prove that

$$|| \beta^{'} + \tau |^2 - | \beta^{'} + \beta_1 + \tau |^2|> \frac{7}{8}\rho^a, \forall\beta^{'} \in A_4, \beta_1 \in \Gamma_\delta(p\rho^{\alpha}). \tag{100}$$

Using $| \beta - \beta^{'} |< \rho^{a-2\alpha}$ (see definition of A_4) by calculations we get

$$| \beta^{'} + \tau |^2 - | \beta^{'} + \beta_1 + \tau |^2= -2(\beta^{'} + \tau, \beta_1)- | \beta_1 |^2=$$

$$-2(\beta + \tau, \beta_1)- | \beta_1 |^2 -2(\beta^{'} - \beta, \beta_1) = -(| \beta + \beta_1 + \tau |^2 - | \beta + \tau |^2) + o(\rho^a).$$

This and (31) imply that (100) and hence (98) holds. Now to estimate the right-hand side of (97) we prove that if $\beta^{'} \in A_4$, $\beta^{''} \in A_4$ and $\beta^{'} \neq \beta^{''}$ then

$$\beta^{'} + \beta_1(\beta^{'}) \neq \beta^{''} + \beta_1(\beta^{''}). \tag{101}$$

Assume the contrary that they are equal. Then we have $\beta^{''} = \beta^{'} + b$, where $b \in \Gamma_\delta(2\rho^{\alpha})$, since $\beta_1(\beta^{'}) \in \Gamma_\delta(\rho^{\alpha})$, $\beta_1(\beta^{''}) \in \Gamma_\delta(\rho^{\alpha})$. It easily follows from the inclusions $\beta^{'} \in A_4$, $\beta^{'} + b \in A_4$ that

$$|| \beta^{'} + \tau |^2 - | \beta^{'} + \tau + b |^2|< \frac{1}{2}\rho^a$$

which contradicts (100)). Thus (101) is proved. Therefore using (98) and Bessel inequality from (97) we obtain the proof of (94) for $i = 2$. Hence (90) is proved for $i = 4$.

Now we prove (90) for $i = 6$. First we note that $A_6 = \{\beta\}$. Indeed if $\beta^{'} \neq \beta$ and $\beta^{'} \in A_6$ then we have $\beta^{'} = \beta + b$, where $b \in \Gamma_\delta(\rho^{\alpha_d})$, and from the relations

$\beta \notin V_b^\delta(\rho^{\frac{1}{2}})$ (see (23) and the definition of S_2), $\mid \beta+\tau \mid = H$, we obtain that $\mid\mid \beta^{'}+\tau\mid^2 - H^2 \mid > \frac{1}{2}\rho^{\frac{1}{2}}$ which contradicts $\beta^{'}+\tau \in R_\delta(H+\rho^{\alpha_d-1})$). Hence

$$C_6 = \sum_{j^{'}\in\mathbb{Z}} (\beta+\tau,\beta+\tau) \mid b(N,j^{'},\beta)\mid^2 = H^2 \sum_{j^{'}\in\mathbb{Z}} \mid b(N,j^{'},\beta)\mid^2 = H^2 \sum_{i=1}^{3} C_{6,i},$$

where $C_{6,1} = \mid b(N,j,\beta)\mid^2$,

$$C_{6,2} = \sum_{\mid j^{'}\delta\mid \geq \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N,j^{'},\beta)\mid^2,\ C_{6,3} = \sum_{\mid j^{'}\delta\mid < \frac{1}{30}\rho^{\frac{a}{2}}, j^{'}\neq j} \mid b(N,j^{'},\beta)\mid^2 .$$

To prove (90) for $i=6$ we show that

$$C_{6,i} = O(\rho^{-2a}),\ \forall i=1,2,3. \tag{102}$$

By (57) this equality holds for $i=1$. For $\mid j^{'}\delta \mid \geq \frac{1}{30}\rho^{\frac{a}{2}}$ the inequality (91) holds. Therefore repeating the proof of (90) for $i=2$ we get the proof of (102) for $i=2$. Arguing as in the proof of (94) for $i=2$ we obtain the proof of (102) for $i=3$. Thus (90) is proved for $i=6$.

Now we prove that

$$C_i \leq \sum_{\beta^{'}\in A_i} (\sum_{j^{'}\in Z} \mid b(N,j^{'},\beta^{'})\mid^2 (H^2 - \frac{1}{3}\rho^{2\alpha_d})) \tag{103}$$

for $i=3,5,7$. Consider the triangle generated by vectors $\beta+\tau$, $\beta^{'}+\tau$, $\beta-\beta^{'}$. For $\beta^{'}\in A_3$ we have

$$H+\rho^{\alpha_d-1} \leq \mid \beta^{'}+\tau \mid \leq H+\frac{1}{9}\rho^{a-1}, \mid \beta-\beta^{'}\mid \geq \rho^{a-2\alpha}.$$

Let θ be the angle between the vectors $\beta+\tau$, and $\beta^{'}+\tau$. If $\mid\theta\mid \leq \frac{\pi}{2}$ then using the cosine theorem we get

$$\mid (\beta+\tau,\beta^{'}+\tau)\mid = \frac{1}{2}(\mid \beta+\tau\mid^2 + \mid \beta^{'}+\tau\mid^2 - \mid \beta-\beta^{'}\mid^2)$$
$$\leq H^2 - \frac{1}{3}\rho^{2a-4\alpha} < H^2 - \frac{1}{3}\rho^{2\alpha_d},$$

since $a-2\alpha > \alpha_d$. Using this and taking into account that $(\beta+\tau,\beta^{'}+\tau) < 0$ for $\frac{\pi}{2} < \mid\theta\mid \leq \pi$ we get the proof of (103) for $i=3$. Similarly if $\beta^{'}\in A_5$, $\mid\theta\mid\leq\frac{\pi}{2}$ then

$$\mid (\beta+\tau,\beta^{'}+\tau)\mid \leq H^2 - \frac{1}{3}\rho^{2\alpha_d}$$

and hence (103) holds for $i=5$. If $\beta^{'}\in A_7$ then $\mid \beta^{'}+\tau\mid \leq H-\rho^{2\alpha_d-1}$ and by (89) we have

$$\mid (\beta+\tau,\beta^{'}+\tau)\mid \leq H^2 - \frac{1}{3}\rho^{2\alpha_d},$$

that is, (103) holds for $i=7$ too. Now (103) and Bessel inequality imply that

$$C_3+C_5+C_7\leq H^2-\frac{1}{3}\rho^{2\alpha_d}=\mid\beta+\tau\mid^2-\frac{1}{3}\rho^{2\alpha_d}.$$

This, (90) and (54) give the proof of Lemma 3, since $2-2a<2\alpha_d$ (see the definition of a in (9)).

The proof of Lemma 4

Let n_1 be a positive integer satisfying the inequality
$\mid(n_1+v)\delta\mid^2\leq 4\rho^{1+\alpha_d}<\mid(n_1+1+v)\delta\mid^2$. Introduce the following sets

$$D_{b',j}(\rho,v,4)=\{x\in H_\delta:\mid 2(x,b^{'})+\mid b^{'}\mid^2+\mid(j+v)\delta\mid^2\mid<4d_\delta\rho^{\alpha_d}\}$$

$$D(\rho,v,4)=\bigcup_{j=-n_1-3}^{n_1}(\bigcup_{b'\in\Gamma_\delta(\rho^{\alpha_d})}D_{b',j}(\rho,v,4),\tag{104}$$

$$S_2^{'}(\rho,b,v)=((V_b^\delta(4\rho^a)\backslash V_b^\delta(\rho^a))\backslash(D(\rho,v,4)\cup D_1(\rho^{\frac{1}{2}})\cup D_2(\rho^{a+2\alpha})))\cap D_3,\tag{105}$$

where

$$D_1(\rho^{\frac{1}{2}})=\bigcup_{b'\in\Gamma_\delta(\rho^{\alpha_d})}V_{b'}^\delta(\rho^{\frac{1}{2}}),\ D_2(\rho^{a+2\alpha})=\bigcup_{b'\in\Gamma_\delta(p\rho^{\alpha})\backslash b\mathbb{R}}V_{b'}^\delta(\rho^{a+2\alpha}),$$

$D_3=(R(\frac{3}{2}\rho-d_\delta-1)\backslash R(\frac{1}{2}\rho+d_\delta+1))$.

Now we prove that the set $S_2^{'}(\rho,b,v)$ contains an element $\beta\in\Gamma_\delta$ satisfying all assertions of Lemma 4. First let us prove that $S_2^{'}(\rho,b,v)\cap\Gamma_\delta$ is nonempty subset of $S_2(\rho)$, that is,

$$S_2^{'}(\rho,b,v)\cap\Gamma_\delta\subset S_2(\rho),\ S_2^{'}(\rho,b,v)\cap\Gamma_\delta\neq\emptyset\tag{106}$$

It follows from the definitions of $S_2^{'}(\rho,b,v)$ and $S_2(\rho)$ (see (23)) that the first relation of (106) holds. To prove the second relation we consider the set

$$D^{'}(\rho)=(V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\backslash(D(\rho,v,6)\cup D_1(2\rho^{\frac{1}{2}})\cup D_2(2\rho^{a+2\alpha})))\cap D_4,$$

where $D_4=R(\frac{3}{2}\rho-1)\backslash R(\frac{1}{2}\rho+1)$. If $\beta+\tau\in D^{'}(\rho)$, where $\beta\in\Gamma_\delta,\tau\in F_\delta$, then one can easily verify that $\beta\in S_2^{'}(\rho,b,v)$. Therefore $\{\ \beta+F_\delta:\beta\in S_2^{'}(\rho,b,v)\cap\Gamma_\delta\}$ is a cover of $D^{'}(\rho)$. Hence

$$\mid S_2^{'}(\rho,b,v)\cap\Gamma_\delta\mid\geq(\mu(F_\delta))^{-1}\mu(D^{'}(\rho)),\tag{107}$$

where $\mid S_2^{'}(\rho,b,v)\cap\Gamma_\delta\mid$ is the number of elements of $S_2^{'}(\rho,b,v)\cap\Gamma_\delta$. Thus to prove the second relation of (106) we need to estimate $\mu(D^{'}(\rho))$. It is not hard to verify that (see Remark 2.1 of [4])

$$\mu((V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\cap D_4)>c_{13}\rho^{d-2+a}.\tag{108}$$

Now we estimate $\mu((V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\cap D_1(2\rho^{\frac{1}{2}})\cap D_4)$. If $b^{'}\in(b\mathbb{R})\cap\Gamma_\delta(\rho^{\alpha_d})$, then one can easily verify that $V_{b^{'}}^\delta(2\rho^{\frac{1}{2}})\cap D_4\subset V_b^\delta(2\rho^a)\cap D_4$. Therefore we need to estimate the measure of $V_b^\delta(3\rho^a)\cap V_{b^{'}}^\delta(2\rho^{\frac{1}{2}})\cap D_4$ for $b^{'}\in\Gamma_\delta(\rho^{\alpha_d})\backslash b\mathbb{R}$. For this we turn the coordinate axes so that the direction of $(1,0,0,...,0)$ coincides with the direction $b^{'}$, that is $b^{'}=(\mid b^{'}\mid,0,0,...,0)$ and the plane generated by $b,b^{'}$ coincides with the plane $(x_1,x_2,0,...0)$, that is, $b=(b_1,b_2,0,...,0)$.Then the condition $x\in V_b^\delta(3\rho^a)\cap V_{b^{'}}^\delta(2\rho^{\frac{1}{2}})\cap D_4$ imply that

$$\begin{aligned} x_1\mid b^{'}\mid&=O(\rho^{\frac{1}{2}}),\\ x_1b_1+x_2b_2&=O(\rho^a).\\ x_1^2+x_2^2+...+x_{d-1}^2&=O(\rho^2).\end{aligned}\tag{109}$$

First equality of (109) shows that $x_1=O(\rho^{\frac{1}{2}})$. Since $b^{'}$ and b are linearly independent vectors of Γ_δ we have $\mid b^{'}\mid\mid b_2\mid\geq\mu(F_\delta)$, where $\mid b^{'}\mid<\rho^{\alpha_d}$. Therefore $\mid b_2\mid\geq\mu(F_\delta)\rho^{-\alpha_d}$ and the second equality of (109) implies that $x_2=O(\rho^{a+\alpha_d})$. Now using the third equality of (109) we obtain that $V_b^\delta(3\rho^a)\cap V_{b^{'}}^\delta(2\rho^{\frac{1}{2}})\cap D_4$ is subset of $[-c_{14}\rho^{\frac{1}{2}},c_{14}\rho^{\frac{1}{2}}]\times[-c_{14}\rho^{a+\alpha_d},c_{14}\rho^{a+\alpha_d}]\times([-c_{14}\rho,c_{14}\rho])^{d-3}$ which has the measure $O(\rho^{d-3+\frac{1}{2}+a+\alpha_d})$. This with $\mid\Gamma_\delta(\rho^{\alpha_d})\mid=O(\rho^{(d-1)\alpha_d})$ give

$$\mu((V_b^\delta(3\rho^a)\cap D_1(2\rho^{\frac{1}{2}})\cap D_4)=O(\rho^{d-3+\frac{1}{2}+a+d\alpha_d})=o(\rho^{d-2+a}),\tag{110}$$

since $d\alpha_d<\frac{1}{2}$ (see the definition of α_d in (7)). In the same way we get

$$\mu((V_b^\delta(3\rho^a)\cap D_2(2\rho^{a+2\alpha})\cap D_4)=O(\rho^{d-3+2a+(d+4)\alpha})=o(\rho^{d-2+a}),\tag{111}$$

since $a+(d+4)\alpha<1$ (see (7) and (9)). To estimate $\mu(D_{b^{'},j}(\rho,v,6))$ we turn the coordinate axes so that the direction of $(1,0,0,...,0)$ coincides with the direction $b^{'}$. Then the condition $x\in D_{b^{'},j}(\rho,v,6)\cap D_4$ imply that

$$2x_1\mid b^{'}\mid+\mid b^{'}\mid^2+\mid(j+v)\delta\mid^2\mid=O(\rho^{\alpha_d}),$$

$$x_1^2+x_2^2+...+x_{d-1}^2=O(\rho^2).$$

These equalities shows that x_1 belongs to the interval of length $O(\rho^{\alpha_d})$ and

$$\mu(D_{b^{'},j}(\rho,v,6)\cap D_4)=O(\rho^{d-2+\alpha_d}).$$

Now using (104) and taking into account that $n_1=O(\rho^{\frac{1}{2}(1+\alpha_d)})$, $\mid\Gamma_\delta(\rho^{\alpha_d})\mid=O(\rho^{(d-1)\alpha_d})$ we obtain

$$\mu(D(\rho,v,4)\cap D_4=O(\rho^{d-2+\frac{1}{2}+(d+\frac{1}{2})\alpha_d})=o(\rho^{d-2+a}),$$

since $a>\frac{1}{2}+(d+\frac{1}{2})\alpha_d$ (see (9) and (7)). This estimation with (110), (111), and (108) implies that $\mu(D^{'}(\rho))>c_{15}\rho^{d-2+a}$. Therefore (107) give the proof of the second equality of (106).

Now take any element β from $S_2^{'}(\rho, b, v) \cap \Gamma_\delta$. It follows from the definitions of of the sets $S_2^{'}(\rho, b, v)$, $D_{b',j}(\rho, v, 4)$, $A(\beta, \rho)$ (see (105) and (27)) that $v \notin A(\beta, \rho)$.

Let us prove the inequalities in (58). By the definition of $S_2^{'}(\rho, b, v)$ we have $\beta \in V_b^\delta(4\rho^a) \backslash V_b^\delta(\rho^a)$. This means that

$$\rho^a \leq \mid 2(\beta, b) + \mid b \mid^2 \mid < 4\rho^a.$$

This with the obvious relations $\mid b \mid = O(1)$, $\mid \tau \mid = O(1)$ imply (58).

Now we prove (59). If $\gamma \in S(\delta, b) \backslash \delta\mathbb{R}$ then

$$\gamma = nb + a\delta, \ n \neq 0, \ n \in \mathbb{Z}, \ a \in \mathbb{R}, \ \mid (\gamma, b) \mid = \mid n \mid \mid b \mid^2 \geq \mid b \mid^2, \tag{112}$$

since each $\gamma \in \Gamma$ has decomposition $\gamma = b^{'} + a\delta$, where $b^{'} \in \Gamma_\delta$, and b is a maximal element of Γ_δ (see (3.2) of [4] and the definition of $S(\delta, b)$ in (13)). This with the relations $(\beta + \tau, \delta) = 0$ give $(\beta + \tau, \gamma) = n(\beta + \tau, b)$. Therefore the first inequality of (58) implies (59).

Let us prove (60). If $\gamma \notin S(\delta, b), \mid \gamma \mid < \mid \rho \mid^\alpha$ then $\gamma = b^{'} + a\delta$, where $a \in \mathbb{R}$, $b^{'} \in \Gamma_\delta(\rho^\alpha) \backslash b\mathbb{R}$, and $(\beta + \tau, \gamma) = (\beta + \tau, b^{'})$. Therefore using $\mid b^{'} \mid = O(\rho^\alpha)$, $\mid \tau \mid = O(1)$ and arguing as in the proof of (58) we see that the relation

$\beta \notin V_{b'}^\delta(\rho^{a+2\alpha})$, (see definition of $S_2^{'}(\rho, b, v)$) implies (60).

The inequality (61) follows from the definition of $f_{\delta, \beta+\tau}(x)$, (59), (60), and from the obvious relation

$$\sum_{\gamma \in \Gamma} \mid \gamma \mid \mid q_\gamma \mid < c_{16}, \forall q(x) \in W_2^s(F, M).$$

The last inequality with (112) imply the convergence of the series (13).

4 APPENDICES

APPENDIX A. THE PROOF OF (76).

Here we estimate the complex conjugate $\overline{C_1(j^{'}, \lambda_{j,\beta})}$ of $C_1(j^{'}, \lambda_{j,\beta})$, namely prove that (see (74))

$$\sum_{(j_1, \beta_1) \in Q(\rho^\alpha, 9r)} \frac{\overline{A(j^{'}, \beta, j^{'} + j_1, \beta + \beta_1)} \, \overline{A(j^{'} + j_1, \beta + \beta_1, j, \beta)}}{\lambda_{j,\beta} - \lambda_{j^{'}+j_1, \beta+\beta_1}} = O(\rho^{-2a} r^2), \tag{A1}$$

where $Q(\rho^\alpha, 9r) = \{(j_1, \beta_1) : \mid j_1\delta \mid < 9r, \ 0 < \mid \beta_1 \mid < \rho^\alpha\}$, $j \in S_1(\rho)$, $\mid j^{'}\delta \mid < r$, $r = O(\rho^{\frac{1}{2}\alpha_2})$. The conditions on indices $j^{'}$, j_1, j and (39) imply that

$\mu_{j^{'}+j_1} = O(r^2)$, $\mu_j = O(r^2)$. These with $\beta \notin V_{\beta_1}^\delta(\rho^a)))$, where $\beta_1 \in \Gamma_\delta(p\rho^\alpha)$, (see (29)) give

$$\lambda_{j,\beta} - \lambda_{j^{'}+j_1, \beta+\beta_1} = -2(\beta, \beta_1) + O(r^2), \ \ \mid (\beta, \beta_1) \mid > \frac{1}{3}\rho^a. \tag{A2}$$

Using this, (34) and (A1) we get

$$\overline{C_1(j^{'}, \lambda_{j,\beta})} = \sum_{\beta_1} \frac{C^{'}}{-2(\beta, \beta_1)} + O(\rho^{-2a} r^2), \tag{A3}$$

where $C^{'}=\sum_{j_1}\overline{A(j^{'},\beta,j^{'}+j_1,\beta+\beta_1)}\overline{A(j^{'}+j_1,\beta+\beta_1,j,\beta)}$. In [4] we proved that (see (3.21), (3.7), Lemma 3.3 of [4])

$$\overline{A(j^{'},\beta,j^{'}+j_1,\beta+\beta_1)}=\sum_{n_1:(n_1,\beta_1)\in\Gamma^{'}(\rho^{\alpha})}c(n_1,\beta_1)a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1),\tag{A4}$$

$$\overline{A(j^{'}+j_1,\beta+\beta_1,j,\beta)}=\sum_{n_2:(n_2,-\beta_1)\in\Gamma^{'}(\rho^{\alpha})}c(n_2,-\beta_1)a(n_2,-\beta_1,j^{'}+j_1,\beta+\beta_1,j,\beta),$$

$$\Gamma^{'}(\rho^{\alpha})=\{(n_1,\beta_1):\beta_1\in\Gamma_{\delta}\backslash 0,n_1\in\mathbb{Z},\beta_1+(n_1-(2\pi)^{-1}(\beta_1,\delta^*))\delta\in\Gamma(\rho^{\alpha})\},$$

$$c(n_1,\beta_1)=q_{\gamma_1},\gamma_1=\beta_1+(n_1-(2\pi)^{-1}(\beta_1,\delta^*))\delta\in\Gamma(\rho^{\alpha}),\tag{A5}$$

$$a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1)=(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j^{'},v(\beta)}(s),\varphi_{j^{'}+j_1,v(\beta+\beta_1)}(s)),$$

$$\begin{aligned}a(n_2,-\beta_1,j^{'}+j_1,\beta+\beta_1,j,\beta)&=(e^{i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j^{'}+j_1,v(\beta+\beta_1)},\varphi_{j,v(\beta)})\\&=(\varphi_{j^{'}+j_1,v(\beta+\beta_1)},e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j,v(\beta)})\\&=\overline{(e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*)s}\varphi_{j,v(\beta)},\varphi_{j^{'}+j_1,v(\beta+\beta_1)})},\end{aligned}\tag{A6}$$

where δ^* is the element of Ω satisfying $(\delta^*,\delta)=2\pi$

Now to estimate the right-hand side of (A3) we prove that

$$\begin{aligned}&\sum_{j_1}a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1)a(n_2,-\beta_1,j^{'}+j_1,\beta+\beta_1,j,\beta)\\&=a(n_1+n_2,0,j^{'},\beta,j,\beta)+O(\rho^{-p\alpha}).\end{aligned}\tag{A7}$$

By definition we have

$$a(n_1+n_2,0,j^{'},\beta,j,\beta)=(e^{i(n_1+n_2)s}\varphi_{j^{'},v(\beta)}(s),\varphi_{j,v(\beta)}(s))=$$

$$(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j^{'},v(\beta)}(s),e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)).$$

This, (A6), and the following formulas

$$\begin{aligned}&e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j^{'},v(\beta)}(s)\\&=\sum_{|j_1\delta|<9r}a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1)\varphi_{j^{'}+j_1,v(\beta+\beta_1)}(s)+O(\rho^{-p\alpha}),\\&e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)\\&=\sum_{|j_1\delta|<9r}\overline{a(n_2,-\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1)}\varphi_{j^{'}+j_1,v(\beta+\beta_1)}+O(\rho^{-p\alpha}),\end{aligned}$$

$$\sum_{j_1}\mid a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1)\mid=O(1)\tag{A8}$$

(see (3.16), (3.17) of [4]) give the proof of (A7). Now from (A7), (A4), (A3) we obtain

$$C^{'}=\sum_{n_1}(\sum_{n_2}(c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j^{'},\beta,j,\beta)+O(\rho^{-p\alpha}))),$$

$$\overline{C_1(j^{'},\lambda_{j,\beta})}=\sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}C_1^{'}(\beta_1,n_1,n_2))+O(\rho^{-2a}r^2),$$

where $C_1^{'}(\beta_1,n_1,n_2)=\dfrac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j^{'},\beta,j,\beta)}{-2(\beta,\beta_1)}$. One can readily verify that

$$C_1^{'}(\beta_1,n_1,n_2)+C_1^{'}(-\beta_1,n_2,n_1)=0. \tag{A9}$$

Therefore $\overline{C_1(j^{'},\lambda_{j,\beta})}=O(\rho^{-2a}r^2)$.

APPENDIX B. THE PROOF OF (47).

Arguing as in the proof of (75) we see that

$$C_2(\Lambda_{j,\beta})=C_2(\lambda_{j,\beta})+O(\rho^{-3a}).$$

Using (A4) we obtain

$$\overline{C_2(\lambda_{j,\beta})}=\sum_{\beta_1,\beta_2}(\sum_{n_1,n_2,n_3}(\sum_{j_1,j_2}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)}{(\lambda_{j,\beta}-\lambda_{j(1),\beta(1)})(\lambda_{j,\beta}-\lambda_{j(2),\beta(2)})}a(n_1,\beta_1,j,\beta,j(1),\beta(1))\times$$
$$a(n_2,\beta_2,j(1),\beta(1),j(2),\beta(2))a(n_3,-\beta_1-\beta_2,j(2),\beta(2),j,\beta),$$

where $(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)$, $(j_2,\beta_2)\in Q(\rho^{\alpha},90r_1)$, $j\in S_1,\beta_1+\beta_2\neq 0$. Applying (A7) two times and using (A8), we get

$$\sum_{j_1}a(n_1,\beta_1,j,\beta,j(1),\beta(1))(\sum_{j_2}a(n_2,\beta_2,j(1),\beta(1),j(2),\beta(2))a(n_3,-\beta_1-\beta_2,j(2),\beta(2),j,\beta))$$

$$=\sum_{j_1}a(n_1,\beta_1,j,\beta,j(1),\beta(1))(a(n_2+n_3,-\beta_1,j(1),\beta(1),j,\beta)+O(\rho^{-p\alpha}))$$

$$=a(n_1+n_2+n_3,0,j,\beta,j,\beta)+O(\rho^{-p\alpha}).$$

Using this in above expression for $C_2(\lambda_{j,\beta})$ and taking into account that

$$\lambda_{j,\beta}-\lambda_{j(1),\beta(1)}=-2(\beta,\beta_1)+O(\rho^{2\alpha_1}),\mid(\beta,\beta_1)\mid>\frac{1}{3}\rho^{a},$$

$$\lambda_{j,\beta}-\lambda_{j(2),\beta(2)}=-2(\beta,\beta_1+\beta_2)+O(\rho^{2\alpha_1}),\mid(\beta,\beta_1+\beta_2)\mid>\frac{1}{3}\rho^{a},$$

which can be proved as (A2), we have $C_2(\lambda_{j,\beta})=O(\rho^{-3a+2\alpha_1})+$

$$\sum_{\beta_1,\beta_2}(\sum_{n_1,n_2,n_3}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)a(n_1+n_2+n_3,0,j,\beta,j,\beta)}{4(\beta,\beta_1)(\beta,\beta_1+\beta_2)}.$$

Grouping in the last sum terms with the equal multiplicands

$$c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2),\ c(n_2,\beta_2)c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2),$$

$$c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2),\ c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1),$$

$$c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1)c(n_2,\beta_2),\ c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2)c(n_1,\beta_1)$$

and using the obvious equality

$$\frac{1}{(\beta,\beta_1)(\beta,\beta_1+\beta_2)}+\frac{1}{(\beta,\beta_2)(\beta,\beta_2+\beta_1)}+\frac{1}{(\beta,\beta_1)(\beta,-\beta_2)}+$$

$$\frac{1}{(\beta,\beta_2)(\beta-,\beta_1)}+\frac{1}{(\beta,-\beta_1-\beta_2)(\beta,-\beta_2)}+\frac{1}{(\beta,-\beta_1-\beta_2)(\beta,-\beta_1)}=0$$

we see that this sum is zero, that is, $C_2(\lambda_{j,\beta})=O(\rho^{-3a+2\alpha_1})$.

APPENDIX C. THE PROOF OF (46).

It follows from (75) that $C_1(\Lambda_{j,\beta})=C_1(\lambda_{j,\beta})+O(\rho^{-3a})$. Therefore we need to prove that

$$\overline{C_1(\lambda_{j,\beta})}=\frac{1}{4}\int_F|f_{\delta,\beta+\tau}(x)|^2\left|\varphi_{j,v}^{\delta}\right|^2dx+O(\rho^{-3a+2\alpha_1}),$$

where

$$\overline{C_1(\lambda_{j,\beta})}\equiv\sum_{\beta_1}(\sum_{j_1}\frac{\overline{A(j,\beta,j+j_1,\beta+\beta_1)}\overline{A(j+j_1,\beta+\beta_1,j,\beta)}}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}},$$

$(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)$, $j\in S_1$, and by (A4)

$$\overline{C_1(\lambda_{j,\beta})}=\sum_{\beta_1}(\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^{\alpha})}(\sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^{\alpha})}(\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}}\times$$

$$a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta).$$

Replacing $\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}$ by $-(2(\beta+\tau,\beta_1)+\mid\beta_1\mid^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))$ and using (A7) for $j^{'}=j$ we have

$$\overline{C_1(j,\lambda_{j,\beta})}=\sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{-2(\beta+\tau,\beta_1)}+$$

$$\sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}(\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)}{2(\beta+\tau,\beta_1)(2(\beta+\tau,\beta_1)+\mid\beta_1\mid^2+\mu_{j+j_1}-\mu_j)}\times$$

$$a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)(\mid\beta_1\mid^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta))).$$

The formula (A9) shows that the first summation of the right-hand side of this equality is zero. Thus we need to estimate the second sum. For this we use the following relation

$$\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)=(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)},T_v\varphi_{j+j_1,v(\beta+\beta_1)})$$

$$= (T_v(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s)$$

$$= ((\mid n_1-(2\pi)^{-1}(\beta_1,\delta^*)\mid^2\mid\delta\mid^2+\mu_j(v))(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}),\varphi_{j+j_1,v(\beta+\beta_1)})$$

$$-2i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))\mid\delta\mid^2(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi'_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s)).$$

Using this, (A7), and the formula

$$\sum_{j_1}(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi'_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s))a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)$$

$$= (e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s))+O(\rho^{-p\alpha}),$$

which can be proved as (A7), we obtain

$$\sum_{j_1}\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)=$$

$$(\mid n_1-(2\pi)^{-1}(\beta_1,\delta^*)\mid^2)\mid\delta\mid^2+\mu_j(v))a(n_1+n_2,0,j,\beta,j,\beta)- \tag{C1}$$
$$2i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))\mid\delta\mid^2(e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s)).$$

Here the last multiplicand can be estimated as follows

$$\mu_j(v)(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s))=(\varphi_{j,v(\beta)}(s),T_v(e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s)))$$

$$=(n_1+n_2)^2\mid\delta\mid^2(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s))+$$

$$2i(n_1+n_2)\mid\delta\mid^2(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s))+\mu_j(v)(\varphi_{j,v(\beta)},e^{i(n_1+n_2)s}\varphi_{j,v(\beta)})$$

and hence

$$(e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s))=\frac{n_1+n_2}{2i}(e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s)).$$

Using this, (C1), and (A7) we get

$$\sum_{j_1}(a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta))\times$$
$$(\mid\beta_1\mid^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))=a(n_1+n_2,0,j,\beta,j,\beta)\times$$
$$(\mid\beta_1\mid^2+\mid n_1-\frac{(\beta_1,\delta^*)}{2\pi}\mid^2\mid\delta\mid^2-(n_1-\frac{(\beta_1,\delta^*)}{2\pi})\mid\delta\mid^2(n_1+n_2))$$
$$=(\mid\beta_1\mid^2+\mid\delta\mid^2(n_1-\frac{(\beta_1,\delta^*)}{2\pi})(-n_2-\frac{(\beta_1,\delta^*)}{2\pi}))a(n_1+n_2,0,j,\beta,j,\beta).$$

Thus $\overline{C_1(j,\lambda_{j,\beta})}=C+O(\rho^{-3a+2\alpha_1})$, where

$$C=\sum_{\beta_1,n_1,n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{4\mid(\beta+\tau,\beta_1)\mid^2}\times \tag{C2}$$
$$(\mid\beta_1\mid^2+(n_1-\frac{(\beta_1,\delta^*)}{2\pi})(-n_2-\frac{(\beta_1,\delta^*)}{2\pi})\mid\delta\mid^2).$$

Now we consider

$$\int_F |f_{\delta,\beta+\tau}(x)|^2 \, |\varphi_{n,v}((\delta,x))|^2 \, dx,$$

where $f_{\delta,\beta+\tau}(x)$ is defined in (9) and by (A5)

$$f_{\delta,\beta+\tau}(x) = \sum_{(n_1,\beta_1)\in\Gamma'_\delta(\rho^\alpha)} \frac{\beta_1 + (n_1 - \frac{(\beta_1,\delta^*)}{2\pi})\delta}{(\beta+\tau,\beta_1)} c(n_1,\beta_1) e^{i(\beta_1+(n_1-\frac{(\beta_1,\delta^*)}{2\pi})\delta,x)}.$$

Here $f_{\delta,\beta+\tau}(x)$ is a vector of $\mathbb{R}^d$ and $|f_{\delta,\beta+\tau}(x)|$ is a norm of this vector. Using $(\beta,\delta)=0$ for $\beta\in\Gamma_\delta$ we obtain

$$|f_{\delta,\beta+\tau}(x)|^2 = \sum_{(n_1,\beta_1),(n_2,\beta_2)\in\Gamma'_\delta(\rho^\alpha)} \frac{(\beta_1,\beta_2) + (n_1 - \frac{(\beta_1,\delta^*)}{2\pi})(n_2 - \frac{(\beta_1,\delta^*)}{2\pi}) \mid \delta \mid^2}{(\beta+\tau,\beta_1)(\beta+\tau,\beta_2)} \times$$

$$c(n_1,\beta_1)c(-n_2,-\beta_2)e^{i(\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}(\beta_1-\beta_2,\delta^*))\delta,x)}.$$

Since $\varphi_{j,v}((\delta,x))$ is a function of (δ,x) we have

$$\int_F e^{i(\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}(\beta_1-\beta_2,\delta^*))\delta,x)} \, |\varphi_{j,v}((\delta,x))|^2 \, dx = 0$$

for $\beta_1 \neq \beta_2$. Therefore

$$\int_F |f_{\delta,\beta+\tau}(x)|^2 \, |\varphi_{j,v}((\delta,x))|^2 \, dx = \sum_{\beta_1,n_1,n_2} \frac{c(n_1,\beta_1)c(-n_2,-\beta_1)}{\mid (\beta+\tau,\beta_1) \mid^2} \times$$

$$(\mid \beta_1 \mid^2 + (n_1 - \frac{(\beta_1,\delta^*)}{2\pi})(n_2 - \frac{(\beta_1,\delta^*)}{2\pi}) \mid \delta \mid^2 a(n_1-n_2,0,j,\beta,j,\beta).$$

Replacing n_2 by $-n_2$ we get

$$\int_F |f_{\delta,\beta+\tau}(x)|^2 \, |\varphi_{n,v}((\delta,x))|^2 \, dx = 4C.$$

(see (C2)). Thus (46) is proved.

APPENDIX D. ASYMPTOTIC FORMULAS FOR $T_v(Q)$.

Let $\mu_0(0) \le \mu_1(0) \le \mu_2(0) \le ...$ be the eigenvalues of the operator $T_0(Q)$. It is well-known that $\mu_{2m+1}(0)$ and $\mu_{2m+2}(0)$ both satisfy

$$\sqrt{\mu} = \mid (m+1)\delta \mid + \frac{1}{16\pi \mid (m+1)\delta \mid^3} \int_F \left| q^\delta(x) \right|^2 dx + O(\frac{1}{m^4})$$

(see [1], page 58). This formula yields the invariant (17). Using the asymptotic formulas for solutions of the Sturm-Liouville equation (see [1], page 63) one can easily obtain that

$$\varphi_{n,v}(s) = e^{i(n+v)s}(1 + \frac{Q_1(s)}{2i(n+v) \mid \delta \mid^2} + \frac{Q(s)-Q(0)-\frac{1}{2}Q_1^2(s)}{4(n+v)^2 \mid \delta \mid^4}) + O(\frac{1}{n^3})),$$

where $Q_1(s) = \int_0^s Q(t)dt$. From this by direct calculations we find $A_0(\zeta)$, $A_1(\zeta)$, $A_2(\zeta)$ (see (14)) and then using these in (15) we get the invariants (16).

Now we consider the eigenfunction $\varphi_{n,v}(s)$ of $T_v(p)$ in case $v \neq 0,\ \frac{1}{2}$ and

$$p(s) = \sum_{m=1,-1} p_m e^{ims}. \tag{D1}$$

The eigenvalues and eigenfunctions of $T_v(0)$ are $(n+v)^2 \mid \delta \mid^2$ and $e^{i(n+v)s}$, for $n \in \mathbb{Z}$. Since the eigenvalues of $T_v(p)$ are simple for $v \neq 0,\ \frac{1}{2}$ by well-known perturbation formula

$$(\varphi_{n,v}(s),\ e^{i(n+v)s})\varphi_{n,v}(s) = e^{i(n+v)s} +$$

$$\sum_{k=1,2,\ldots} \frac{(-1)^{k+1}}{2i\pi} \int\limits_C (T_v(0)-\lambda)^{-1} p(x))^k (T_v(0)-\lambda)^{-1} e^{i(n+v)s} d\lambda, \tag{D2}$$

where C is a contour containing only the eigenvalue $(n+t)^2 \mid \delta \mid^2$. Using

$$(T_v(0)-\lambda)^{-1} e^{i(n+v)s} = \frac{e^{i(n+v)s}}{(n+v)^2 \mid \delta \mid^2 - \lambda}$$

and (D1) we see that the k-th ($k = 1, 2, 3, 4$) term F_k of the series (D2) has the form

$$F_1 = \frac{1}{2i\pi} \int\limits_C \sum_{m=1,-1} \frac{p_m e^{i(n+m+v)s}}{((n+v)^2 \mid \delta \mid^2 - \lambda)((n+m+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda,$$

$$F_2 = \frac{-1}{2i\pi} \int\limits_C \sum_{m,l=1,-1} \frac{p_m p_l e^{i(n+m+l+v)s}}{((n+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda,$$

$$F_3 = \frac{1}{2i\pi} \int\limits_C \sum_{m,l,k=1,-1} \frac{p_m p_l p_k e^{i(n+m+l+k+v)s}}{((n+v)^2 \mid \delta \mid^2 - \lambda)((n+m+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+k+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda,$$

$$F_4 = \frac{-1}{2i\pi} \int\limits_C \sum_{m,l,k,r=1,-1} \frac{p_m p_l p_k p_r e^{i(n+m+l+k+r+v)s}}{((n+m+l+k+r+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+l+k+v)^2 \mid \delta \mid^2 - \lambda)((n+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda.$$

Since the distance between $(n+v)^2 \mid \delta \mid^2$ and $(n^{'}+v)^2 \mid \delta \mid^2$ for $n^{'} \neq n$ is greater than $c_{17}n$, we can choose the contour C such that

$$\frac{1}{\mid (n^{'}+v)^2 \mid \delta \mid^2 - \lambda \mid} < \frac{c_{18}}{n},\ \forall \lambda \in C,\ \forall n^{'} \neq n$$

and the length of C is less than c_{19}. Therefore

$$\sum_{k=5,6,\ldots} F_k = O(\frac{1}{n^5}).$$

Now we calculate the integrals in $F_1,\ F_2,\ F_3,\ F_4$ by Cauchy integral formula and then decompose the obtained expression in power of $\frac{1}{n}$. Then

$$F_1 = e^{i(n+v)s}((p_1e^{is} - p_{-1}e^{-is})\frac{1}{\mid \delta \mid^2}(\frac{-1}{2n} + \frac{v}{2n^2} - \frac{4v^2+1}{8n^3} + O(\frac{1}{n^4}))+$$

$$(p_1e^{is} + p_{-1}e^{-is})\frac{1}{\mid \delta \mid^2}(\frac{v}{4n^2} - \frac{v}{2n^3} + \frac{12v^2+1}{16n^4} + O(\frac{1}{n^5}))).$$

Let $F_{2,1}$ and $F_{2,2}$ be the sum of terms in F_2 for which $m+l=\pm 2$ and $m+l=0$ respectively, i.e., $F_2 = F_{2,1} + F_{2,2}$, where

$$F_{2,1} = e^{i(n+v)s}(((p_1)^2e^{2is} + (p_{-1})^2e^{-2is})\frac{1}{\mid \delta \mid^4}(\frac{-1}{8n^2} + \frac{-v}{4n^3} - \frac{12v^2+7}{32n^4} + O(\frac{1}{n^5}))+$$

$$((p_1)^2e^{2is} - (p_{-1})^2e^{-2is})\frac{1}{\mid \delta \mid^4}(\frac{-3}{16n^3} + O(\frac{1}{n^4}))),$$

$$F_{2,2} = e^{i(n+v)s}\left|p_1\right|^2(\frac{c_{20}}{n^2} + + \frac{c_{21}}{n^3} + \frac{c_{22}}{n^4} + O(\frac{1}{n^5}))$$

and c_{20}, c_{21}, c_{22} are known constants. Similarly $F_3 = F_{3,1} + F_{3,2}$, where $F_{3,1}$ and $F_{3,2}$ are the sum of terms in F_3 for which $m+l+k=\pm 3$ and $m+l+k=\pm 1$ respectively. Hence

$$F_{3,1} = e^{i(n+v)s}((p_1^3e^{3is} - p_{-1}^3e^{-is})\frac{1}{\mid \delta \mid^6}(\frac{-1}{48n^3} + O(\frac{1}{n^4}))+$$

$$(p_1^3e^{3is} + p_{-1}^3e^{-3is})\frac{1}{\mid \delta \mid^6}(\frac{1}{16n^4} + O(\frac{1}{n^5}))),$$

$$F_{3,2} = e^{i(n+v)s}((p_1e^{is} - p_{-1}e^{-is})\left|p_1\right|^2(\frac{c_{23}}{n^3} + \frac{c_{24}}{n^4} + O(\frac{1}{n^5}))+$$

$$(p_1e^{is} + p_{-1}e^{-is})\left|p_1\right|^2(\frac{c_{25}}{n^4} + O(\frac{1}{n^5}))).$$

In the same way we can write $F_4 = F_{4,1} + F_{4,2} + F_{4,3}$, where $F_{4,1},\ F_{4,2},\ F_{4,3}$ are the sum of terms in F_4 for which $m+l+k+r=\pm 4$, $m+l+k+r=\pm 2$, $m+l+k+r=0$ respectively. Thus

$$F_{4,1} = e^{i(n+v)s}(p_1^4e^{4is} + p_{-1}^4e^{-4is})\frac{1}{\mid \delta \mid^8}(\frac{1}{384n^4} + O(\frac{1}{n^5})),$$

$$F_{4,2}=e^{i(n+v)s}(p_1^2e^{2is}+p_{-1}^2e^{-2is})\,|p_1|^2\,(\frac{c_{26}}{n^4}+O(\frac{1}{n^5}))),$$

$$F_{4,3}=e^{i(n+v)s}\,|p_1|^4\,(\frac{c_{27}}{n^4}+O(\frac{1}{n^5}))).$$

Since $p_{-1}^ke^{-iks}$ is conjugate of $p_1^ke^{iks}$ the real and imaginary parts of $F_ke^{-i(n+v)s}$ consist of terms with multiplicands $p_1^ke^{iks}+p_{-1}^ke^{-kis}$ and $p_1^ke^{iks}-p_{-1}^ke^{-iks}$ respectively. Taking into account this and using the above estimations we get

$$|(\varphi_{n,v}(s),e^{i(n+v)s})\varphi_{n,v}(s)|^2=2(\sum_{k=1,2,3,4}\mathbf{Re}(F_k)+\mathbf{Re}(F_1F_2)+\mathbf{Re}(F_1F_3))$$
$$+|F_1|^2+|F_2|^2+O(n^{-5})=$$

$$1+\frac{1}{2n^2}\frac{1}{|\,\delta\,|^2}(p_1e^{is}+p_{-1}e^{-is}+c_{28}|p_1|^2)+\frac{1}{n^3}((p_1e^{is}+p_{-1}e^{-is})c_{29}$$
$$+c_{30}|p_1|^2)+\frac{1}{n^4}((p_1e^{is}+p_{-1}e^{-is})c_{31}+c_{32}|p_1|^2+c_{33}|p_1|^4$$
$$+c_{34}|p_1|^2(p_1e^{is}+p_{-1}e^{-is})+(c_{35}+c_{36}|p_1|^2)(p_1^2e^{2is}+p_{-1}^2e^{-2is}))+O(\frac{1}{n^5}),$$

where $\mathbf{Re}(F)$ denotes the real part of F. On the other hand

$$|(\varphi_{n,v}(s),e^{i(n+v)s})|^2=(c_{37}\frac{1}{n^2}+c_{38}\frac{1}{n^3}+c_{39}\frac{1}{n^4})|p_1|^2+c_{40}\frac{1}{n^4}|p_1|^4+O(\frac{1}{n^5}).$$

The formula (19) follows from these equalities and (20) is a consequence of (19), (17) and (15) for $k=2,4$ ■

References

[1] Eastham, M.S.P., "The Spectral Theory of Periodic Differential Equations", Scotting Academic Press (1973).

[2] Eskin, G., Ralston, J., Trubowitz, E., "On Isospectral Periodic Potential in R^n. II" *Comm. on Pure and Appl. Math.* **37**, 715-753 (1984).

[3] Feldman,J., Knorrer, H., Trubowitz, E., "The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator", Comment. Math. Helvetica, 66, 557-579 (1991).

[4] Veliev, O. A., "Perturbation theory for the periodic multidimensional Schrödinger operator and the Bethe-Sommerfeld conjecture", International Journal of Contemporary Math. Sciences, Vol.2, no.2, 19-87 (2007).